A Discontinuous Problem with Quasilinear Operator

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November 2000

1 Introduction

The principal reason to study PDE involving discontinuous nonlinearities (DNDE) is due to many free boundary problems which can be reduced to boundary value problems of DNDE (cf. [Chang1], [Chang2]). In addition, sometimes, it is useful to put the original PDE problem into a large category, for instance, DNDE ([Chang2]).

This paper is about a class of elliptic equations where the nonlinearity is discontinuous. These equations serve as models in Mathematical Physics problems; for instance, this type of equations can appear in phenomenons related with filtration of non newtonian fluids in porous mediums.

In this work we extend results obtained by professors Arcoya and Calahorrano, [Arco-Cala].

In concrete, given $\Omega \subset \mathbb{R}^N$, bounded domain, we consider the problem

$$\left\{ \begin{array}{rl} -\Delta_p u = h(x)f(u) + q(x), & in \quad \Omega \\ u = 0, & on \quad \partial\Omega \end{array} \right\}; \tag{1}$$

where $q \in L^{p'}(\Omega)$ and f is a discontinuous nonlinearity which is assumed with only one upward discontinuity; i.e., $\exists! a \in \mathbb{R}$ such that $f \in C(\mathbb{R} - \{a\}, \mathbb{R})$, and $f(a) \in [f(a-), f(a+)]$; where $f(a\pm) = \lim_{\varepsilon \to 0^+} f(a \pm \varepsilon)$ and $f(a-) < f(a+) < \infty$.

Because of the last condition the associated functional, $I : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$I(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \int_\Omega q(x)u(x)dx - \int_\Omega \int_0^{u(x)} f(t)h(x)dtdx,$$

is not Fréchet differentiable and, therefore, the usual critical point theory is not applicable.

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Then, our objective is find good hipothesis on the functions q, f, h which permit us to prove existence of weak solutions for (1). In this way, we use the critical point theory for locally Lipschitz functionals developed by Chang, [Chang], and the Clarke's generalized gradient, [Clarke]

Because f is discontinuous, we consider two concepts of solutions for (1). In the first one, we say that a function $u \in W_0^{1,p}(\Omega)$ is a solution for the multivalued problem associated to (1), if u satisfies

$$-\Delta_p u(x) - q(x) \in h(x)\hat{f}(u(x)), \quad a.e.\Omega \tag{2}$$

where $\hat{f}$ is the multivalued function given by

$$\hat{f}(s) = \left\{ \begin{array}{cc} \{f(s)\}, & if \quad s \neq a \\ \left[f(a-), f(a+)\right], & if \quad s = a \end{array} \right\}.$$

However, there exists a second, more restictive (but more interesting), criterion of solution. We say that $u \in W_0^{1,p}(\Omega)$ is a solution for (1) provided

$$-\Delta_p u(x) = h(x)f(u(x)) + q(x), \quad a.e.\Omega. \tag{3}$$

Clearly, such a solution is also a solution in the former sense.

2 Background

Professors Ambrosetti and Badiale, [Ambro-Ba], studied the semilinear elliptic problem

$$\left\{ \begin{array}{cc} -\Delta u = f(u) + q(x), & in \quad \Omega \\ u = 0, & on \quad \partial\Omega \end{array} \right\} \tag{4}$$

where $q \in L^2(\Omega)$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ verifies

(f1) There exists $a \in \mathbb{R}$ such that $f \in C(\mathbb{R} - \{a\}, \mathbb{R})$ and $f(a) \in [f(a-), f(a+)]$ with $f(a-) < f(a+) < \infty$.

Clearly, the problem (4) is a particular case of (1); where $p = 2$ and $h(x) = 1, \forall x \in \Omega$.

As it was said, the dificulty is that the associated Euler functional $I : H_0^1(\Omega) \longrightarrow \mathbb{R}$, given by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega \int_0^{u(x)} f(u(x))dx - \int_\Omega q(x)u(x)dx,$$

is not Fréchet differentiable. Hence, professors Ambrosetti and Badiale used the Clarke dual principle to obtain a dual functional $\Phi \in C^1(L^2(\Omega))$ such that its critical points u are solutions of the multivalued problem associated to (4). They also showed that if $-q(x) \notin [f(a-), f(a+)], \quad a.e.\Omega$, or if u is a local minimum of Φ, then u satisfies that $|\{x \in \Omega : u(x) = a\}| = 0$, and therefore u is solution of

$$-\Delta u(x) = f(u(x)) + q(x), \quad a.e.\Omega.$$

The previous one motived to professors Arcoya and Calahorrano, [Arco-Cala], to generalize these results for the p-Laplacian version of (4), i.e., they considered the problem

$$\left\{ \begin{array}{rl} -\Delta_p u = f(u) + q(x), & in \quad \Omega \\ u = 0, & on \quad \partial\Omega \end{array} \right\} \tag{5}$$

where, $q \in L^{p'}(\Omega)$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies (f1).

Once in this case it does not seem easy to apply the dual line of reasoning, professors Arcoya and Calahorrano considered the condition

(f2) There exists $\sigma > 0$ such that $|f(s)| \leq C_1 + C_2|s|^\sigma, \quad \forall s \in \mathbb{R}$, where $p \leq 1 + \sigma < p^*$ and

$$p^* = \left\{ \begin{array}{ll} \frac{Np}{N-p}, & if \quad p < N, \\ +\infty, & if \quad p \geq N. \end{array} \right\},$$

which permits to prove that the associated functional is locally Lipschitz continuous with generalized gradient $\partial I(u)$, in each $u \in W_0^{1,p}(\Omega)$, [Chang]. Then, professors Arcoya and Calahorrano showed that

(a) $u \in W_0^{1,p}(\Omega)$ is a critical point of I [1] if and only if

$$-\Delta_p u(x) - q(x) \in \hat{f}(u(x)), \quad a.e.\Omega.$$

(b) If $-q(x) \notin [f(a-), f(a+)] \quad a.e.\Omega$ and u is a critical point of I, it verifies

$$|\{x \in \Omega| \quad u(x) = a\}| = 0, \tag{6}$$

and, therefore, u satisfies

$$-\Delta_p u(x) = q(x) + f(u(x)), \quad a.e.\Omega. \tag{7}$$

(c) If u is a local minimum of I then, (6) and (7) likewise hold.

3 Principal result

Given $\Omega \subset \mathbb{R}^N$, bounded domain, we study the problem

$$\left\{ \begin{array}{rl} -\Delta_p u = h(x)f(u) + q(x), & in \quad \Omega \\ u = 0, & on \quad \partial\Omega \end{array} \right\} \tag{8}$$

where, Δ_p is the p-Laplacian operator defined by

$$\Delta_p u = div\{|\nabla u|^{p-2}\nabla u\}, \quad 1 < p < \infty.$$

We assume $q \in L^{p'}(\Omega)$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ verifying

(f1) There exists only one $a \in \mathbb{R}$ such that $f \in C(\mathbb{R} - \{a\}, \mathbb{R})$, and $f(a) \in [f(a-), f(a+)]$, where $f(a-) < f(a+) < \infty$.

In addition , for h we suppose

(h1) $0 < m := \inf(h(\Omega)) \leq \sup(h(\Omega)) =: M$

[1] In this case $u \in W_0^{1,p}(\Omega)$ is a critical point of I when $0 \in \partial I(u)$.

From the condition (h1), it is clear that $h \in L^\infty(\Omega)$.

Now we consider the functional $I : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega q(x)u(x)dx - \int_\Omega F(u(x))h(x)dx, \tag{9}$$

where $F : [0, \infty[\longrightarrow \mathbb{R}$ is given by

$$F(t) = \int_0^t f(s)ds.$$

The next result, [Mayor], shows the relationship between the functional I and the problem (8).

Lema 3.1 *The functional $I : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined in (9) has as Euler equation*

$$-\Delta_p u = h(x)f(u) + q(x),$$

i.e., the weak derivative of I, in the point $u \in W_0^{1,p}(\Omega)$, is given by

$$I'_G(u)v = \int_\Omega [-\Delta_p u - h(x)f(u) - q(x)]v(x)dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

It is clear, from the discontinuity of f, that I is not Fréchet diferenciable. However, the following hipothesis is very useful.

(f2) There exists $\sigma > 0$ such that

$$|f(s)| \leq C_1 + C_2|s|^\sigma, \quad \forall s \in \mathbb{R};$$

where

$$p \leq 1 + \sigma \leq p^*$$

and

$$p^* = \left\{ \begin{array}{lll} \frac{Np}{N-p}, & si & p < N \\ +\infty, & si & p \geq N \end{array} \right\}.$$

Note 3.1 *From the condition (f2), it verifies, [Chang, p.107], that the function $\Phi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by*

$$\Phi(x, s) = h(x)F(t),$$

and the functional $N : L^{1+\sigma}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$N(u) = \int_\Omega h(x)F(u(x))dx,$$

are locally Lipschitz continuous.

Because of the last observation, it is clear that the functional I is likewise locally Lipschitz and, therefore, [Chang, p.103,104], I has in each point $u \in W_0^{1,p}(\Omega)$ a generalized gradient $\partial I(u)$.

Note 3.2 *We say that $u \in W_0^{1,p}(\Omega)$ is a critical point of I, if $0 \in \partial I(u)$.*

The next one is our principal result.

Theorem 3.1 *If the conditions (f1), (f2) and (h1) hold, then*

(1) $u \in W_0^{1,p}(\Omega)$ *is a critical point of I if and only if*

$$-\Delta_p u(x) - q(x) \in h(x)\hat{f}(u(x)), \quad a.e.\Omega,$$

where $\hat{f}$ *is the multivalued function*

$$\hat{f}(s) = \left\{ \begin{array}{ll} \{f(s)\}, & s \neq a \\ \left[f(a-), f(a+)\right], & s = a \end{array} \right\}.$$

(2) *If* $u \in W_0^{1,p}(\Omega)$ *is a critical point of I and*

$$-q(x) \notin [\alpha^-, \alpha^+], \quad a.e.\Omega,$$

where

$$\alpha^- = \min\{mf(a-), Mf(a-)\},$$
$$\alpha^+ = \max\{mf(a+), Mf(a+)\},$$

then

$$|\{x \in \Omega | u(x) = a\}| = 0 \tag{10}$$

and, hence, $u \in W_0^{1,p}(\Omega)$ *satisfies*

$$-\Delta_p u(x) = q(x) + h(x)f(u(x)), \quad a.e.\Omega. \tag{11}$$

(3) *If* $u \in W_0^{1,p}(\Omega)$ *is a local minimum of I, then (10) and (11) likewise hold.*

PROOF

We consider well known the tools of [Chang].

(1) For $u \in W_0^{1,p}(\Omega)$, the generalized gradient is given by

$$\partial I(u) = \{Au\} - \partial J(u) + \{Bu\};$$

where $A, B, J : W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ are the functionals defined by

$$< Au, v >= \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, dx, \quad \forall v \in W_0^{1,p}(\Omega),$$

$$< Bu, v >= -\int_\Omega q(x)v(x) \, dx, \quad \forall v \in W_0^{1,p}(\Omega),$$

$$< Ju, v >= \int_\Omega h(x)F(u(x))v(x) \, dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

Moreover, we have $\partial J(u) \subset [h(\cdot)f(u-), h(\cdot)f(u+)]$.

In this way, $u \in W_0^{1,p}(\Omega)$ is a critical point of I iff there exists $w \in \partial J(u)$ such that

$$Au + Bu = w,$$

and

$$w \in h(x)\hat{f}(u(x)), \quad a.e.\Omega.$$

Observe that if $w - Bu \in (L^{1+\sigma}(\Omega))' \subset W^{-1,p'}(\Omega)$ then, $Au \in L^{(1+\sigma)/\sigma}(\Omega)$. But if $L^{p'}(\Omega) \subset (L^{1+\sigma}(\Omega))' \subset W^{-1,p'}(\Omega)$, then we have

$$< Au, v >=< w - Bu, v >, \quad \forall v \in W_0^{1,p}(\Omega);$$

i.e.,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx = \int_\Omega q(x)v(x)dx + \int_\Omega w(x)v(x)dx, \quad \forall v \in W_0^{1,p}(\Omega);$$

and according with this,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx = \int_\Omega (q(x) + w(x))v(x)dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

It follows that

$$-\Delta_p u = w + q \in L^{(1+\sigma)/\sigma}(\Omega), \quad a.e.\Omega$$

and

$$-\Delta_p u - q(x) \in h(x)\hat{f}(u(x)), \quad a.e.\Omega.$$

(2) Suppose that $u \in W_0^{1,p}(\Omega)$ is a critical point of I.

Let Γ be the level set:

$$\Gamma = \{x \in \Omega | u(x) = a\}.$$

From the part (1) of the proof we have

$$-\Delta_p u(x) - q(x) \in h(x)[f(a-), f(a+)], \quad a.e.\Gamma;$$

and, for the definition of α^- and α^+, it follows

$$-\Delta_p u(x) - q(x) \in [\alpha^-, \alpha^+], \quad a.e.\Gamma.$$

Now, using a Morrey—Stampacchia theorem, [Morrey, Th. 3.2.2, p.69], we have

$$-\Delta_p u(x) = 0, \quad a.e.\Gamma,$$

and

$$-q(x) \in [\alpha^-, \alpha^+], \quad a.e.\Gamma.$$

In this way, if $-q(x) \notin [\alpha^-, \alpha^+], \quad a.e.\Omega$, then, $|\Gamma| = 0$.

From the part (1) of the proff, it is likewise clear that

$$-\Delta_p u(x) - q(x) = h(x)f(u(x)), \quad a.e.\Omega - \Gamma;$$

and from this,

$$-\Delta_p u(x) - q(x) = f(u(x)), \quad a.e.\Omega.$$

(3) Suppose that $u \in W_0^{1,p}(\Omega)$ is a local minimum of I. Using a similar argument, as the one of the part (2) of the proof, we show that $-q(x) \in [\alpha^-, \alpha^+], \quad a.e.\Gamma$.

Let $\psi \in W_0^{1,p}(\Omega)$ be a positive continuous function. From the hipothesis of minimum of $u \in W_0^{1,p}(\Omega)$, there exists $\delta > 0$ such that

$$I(u + \epsilon\psi) - I(u) \geq 0, \quad \forall |\epsilon| \leq \delta. \tag{12}$$

It is evident that for all $\epsilon \neq 0$, we have

$$\frac{I(u+\epsilon\psi)-I(u)}{\epsilon} = \frac{1}{p}\int_\Omega \frac{|\nabla u+\epsilon\nabla\psi|^p - |\nabla u|^p}{\epsilon}dx-$$

$$-\int_\Omega \frac{F(u+\epsilon\psi)-F(u)}{\epsilon}h(x)dx - \int_\Omega q(x)\psi(x).$$

Now we will prove that $|\Gamma| = 0$.

(a) First, we should prove that $|\{x \in \Gamma| - q(x) \neq \alpha^+\}| = 0$. To do it, suppose the opposite; e.d.,

$$|\{x \in \Gamma| - q(x) \neq \alpha^+\}| > 0.$$

Then, because of the Lebesgue dominated convergence and (12), we have:

$$0 \leq \lim_{\epsilon\to 0^+} \frac{I(u+\epsilon\psi)-I(u)}{\epsilon} =$$

$$= \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\psi dx - \int_\Omega f(u+)\psi(x)dx - \int_\Omega q(x)\psi(x)dx$$

$$= -\int_\Omega [\Delta_p u + h(x)f(u+) + q(x)]\psi(x)dx$$

$$= -\int_\Gamma [\Delta_p u + h(x)f(u+) + q(x)]\psi(x)dx$$

$$= -\int_\Gamma [\alpha^+ + q(x)]\psi(x)dx;$$

and, how we have that $q(x) \leq \alpha^+, a.e.\Gamma$ and $|\{x \in \Gamma| - q(x) < \alpha^+\}| > 0$, it follows the contradiction

$$0 \leq -\int_\Gamma [\alpha^+ + q(x)]\psi(x)dx < 0.$$

(b) In similar form, it is proved that $|\{x \in \Gamma| - q(x) \neq \alpha^-\}| = 0$.

How $\Gamma = \{x \in \Gamma| - q(x) \neq \alpha^+\} \cup \{x \in \Gamma| - q(x) \neq \alpha^-\}$, it follows, from (a) and (b), that $|\Gamma| = 0$.

■

4 Application

Here, we present a simple application of the before theorem.

We consider the problem (8) with h satisfying (h1) and the nonlinearity f satisfying (f1) and the following condition

(f2') There exist $\alpha, \beta > 0$ such that

$$f(s) \leq \alpha|s|^{p-1} + \beta, \quad \forall s \in \mathbb{R},$$

with

$$\alpha < \frac{\lambda_1}{M};$$

where λ_1 is the first eigenvalue of $-\Delta_p$.

We have the following result

Theorem 4.1 *The problem (8), where f verifies (f1) and (f2'), has at least one solution $u \in W_0^{1,p}(\Omega)$ satisfying*

$$-\Delta_p u(x) = q(x) + h(x)f(u(x)), \quad a.e.\Omega,$$

with

$$|\{x \in \Omega | u(x) = a\}| = 0.$$

PROOF

By the characteristic of λ_1, [Ana], we have

$$\lambda_1 = \inf \left\{ \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} : \ u \in W_0^{1,p}(\Omega) - \{0\} \right\};$$

which implies

$$\int_\Omega |\nabla u|^p dx \geq \lambda_1 \int_\Omega |u|^p dx, \quad \forall u \in W_0^{1,p}(\Omega).$$

Using this and (f2'), we have for $u \in W_0^{1,p}(\Omega)$

$$I(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \int_\Omega h(x)F(u(x))dx - \int_\Omega q(x)u(x)dx$$

$$\geq \frac{1}{p} ||\nabla u||^p_{L^p(\Omega)} dx - ||q||_{L^{p'}(\Omega)} ||u||_{L^p(\Omega)} - M \int_\Omega |F(u(x))| dx;$$

which implies

$$I(u) \geq \frac{1}{p} ||\nabla u||^p_{L^p(\Omega)} dx - ||q||_{L^{p'}(\Omega)} ||u||_{L^p(\Omega)} - \frac{M\alpha}{p\lambda_1} \int_\Omega |\nabla u|^p dx.$$

Then, we have proved that

$$I(u) \geq \frac{1 - (\alpha M)/\lambda_1}{p} ||\nabla u||^p_{L^p(\Omega)} - k||u||_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega);$$

where $k \in \mathbb{R}$. From the last one and considering that $0 < \alpha < \lambda_1/M$ implies $1 - (\alpha M)/\lambda_1 > 0$, it follows that I is coercive.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$. It is clear that

$$I(u) \leq \liminf_n \ f(u_n).$$

Then, the functional I is coercive and weakly lower semicontinuous. From this, it follows that there exists $u \in W_0^{1,p}(\Omega)$, local minimum of I. We conclude with the part (3) of the theorem (3.1).

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