

REMARKS ON THE SPATIAL ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A 1D MODEL OF EQUATORIAL OCEANIC FLOWS

MANUEL FERNANDO CORTEZ

Departamento de Matemáticas
Escuela Politécnica Nacional
Ladrón de Guevara E11-253, Quito, Ecuador

OSCAR JARRÍN*

Escuela de Ciencias Físicas y Matemáticas
Universidad de Las Américas
Vía a Nayón, C.P.170124, Quito, Ecuador

ABSTRACT. We consider a new nonlocal and nonlinear one-dimensional evolution model arising in the study of oceanic flows in equatorial regions, recently derived in [A. Constantin and L. Molinet, Global Existence and Finite-Time Blow-Up for a Nonlinear Nonlocal Evolution Equation, Commun. Math. Phys. 402 (2023), 3233–3252].

We investigate the spatial asymptotic behavior of its solutions. In particular, we observe the influence of the Coriolis effect, which, even for rapidly decaying initial data, yields solutions that decay at the rate $1/|x|$. Thereafter, we shed light on the optimality of this decay rate.

1. INTRODUCTION

Introduction to the model. Very recently, in [7], P. Constantin and L. Molinet derived a new nonlocal and nonlinear one-dimensional evolution model arising in the study of *oceanic flows in equatorial regions*. In what follows, we provide a brief outline of the derivation of this model. For a more detailed exposition, see Section 2 of [7].

Within the framework of the physical coordinates - where the x -axis points horizontally due east, the y -axis points horizontally due north, and the z -axis pointing vertically - the starting point are the Navier–Stokes system governing equatorial ocean dynamics on the f -plane, *i.e.*, where the Coriolis parameter is approximated as a constant:

$$\begin{cases} \partial_t u + u \partial_x u + v \partial_y u + w \partial_z u - \beta w = -\frac{1}{\rho} \partial_x P + \mu_1 (\partial_x^2 u + \partial_y^2 u) + \mu_2 \partial_z^2 u, \\ \partial_t v + u \partial_x v + v \partial_y v + w \partial_z v = -\frac{1}{\rho} \partial_y P + \mu_1 (\partial_x^2 u + \partial_y^2 u) + \mu_2 \partial_z^2 u, \\ \partial_t w + u \partial_x w + v \partial_y w + w \partial_z w + \beta w = -\frac{1}{\rho} \partial_w P - g + \mu_1 (\partial_x^2 w + \partial_y^2 w) + \mu_2 \partial_z^2 w, \\ \partial_x u + \partial_y v + \partial_z w = 0. \end{cases}$$

Here, $u = u(t, x, y, z)$, $v = v(t, x, y, z)$ and $w = w(t, x, y, z)$ denote the components of the fluid velocity in the directions of azimuth, latitude, and elevation, respectively, and $P = P(t, x, y, z)$ is the pressure. Additionally, $\rho > 0$ is the constant water density, $g > 0$ is the Earth’s gravitational acceleration at the surface, $\mu_1, \mu_2 > 0$ are the horizontal and vertical viscosity coefficients, and $\beta \neq 0$ is a parameter characterizing the Coriolis effect. Finally, the equation $\partial_x u + \partial_y v + \partial_z w = 0$ describes the mass conservation.

1991 *Mathematics Subject Classification.* Primary: 35B40; Secondary: 35B30, 35A01.

Key words and phrases. Models of Equatorial Oceanic Flows; Coriolis effect; Hilbert transform; Spatial asymptotics of solutions; Blow-up criterion.

*Corresponding author: Oscar Jarrín.

Exploiting the confinement of *equatorial flows*, the meridional velocity component v , as well as the y -axis can be neglected, yielding the system for $u = u(t, x, z)$ and $w = w(t, x, z)$:

$$\begin{cases} \partial_t u + u \partial_x u + w \partial_z u - \beta w = -\frac{1}{\rho} \partial_x P + \mu_1 \partial_x^2 u + \mu_2 \partial_z^2 u, \\ \partial_t w + u \partial_x w + w \partial_z w + \beta w = -\frac{1}{\rho} \partial_w P - g + \mu_1 \partial_x^2 w + \mu_2 \partial_z^2 w, \\ \partial_x u + \partial_z w = 0. \end{cases}$$

Thereafter, by restricting the motion to a fixed depth $z = z_0$ and using the incompressibility constraint, a harmonic stream function is introduced to rigorously deduce that the vertical velocity w is related to the horizontal velocity u by

$$w(t, x, z_0) = -\mathcal{H}u(t, x, z_0),$$

where $\mathcal{H}$ denotes the Hilbert transform (for an explicit definition, see expression (2) below). Additionally, it is deduced that the horizontal pressure gradient acts as an external forcing term

$$-\frac{1}{\rho} \partial_x P(t, x, z_0) := f(t, x).$$

Finally, by setting a unified viscosity constant

$$\mu := \mu_1 - \mu_2 > 0,$$

the authors of [7] show that the previous system reduces to the following one-dimensional nonlocal nonlinear model:

$$\partial_t u + u \partial_x u + (\mathcal{H}u) \partial_x (\mathcal{H}u) + \beta \mathcal{H}u - \mu \partial_x^2 u = f, \quad \mu > 0, \quad \beta \neq 0,$$

which describes the evolution of the *horizontal velocity* $u(t, x)$ in the context of *equatorial oceanic flows*.

Previous theoretical results. The initial value problem associated with this model is also studied in [7] under the periodic spatial condition $x \in \mathbb{T} := \mathbb{R} \setminus \mathbb{Z}$. Specifically, for $s \in \mathbb{R}$, we denote by $H_0^s(\mathbb{T})$ the closed subspace of zero-mean functions in the Sobolev space $H^s(\mathbb{T})$. It is then shown that the model above is locally well-posed in the space $H_0^s(\mathbb{T})$ for $s > -\frac{1}{2}$, provided that the initial datum and the external source term satisfy

$$u_0 \in H_0^s(\mathbb{T}), \quad f \in L^\infty([0, +\infty[, H_0^{s'}(\mathbb{T})), \quad s' \geq s.$$

Moreover, taking $f = 0$ and using similar arguments as in [1, 11], it follows that $H_0^{-\frac{1}{2}}(\mathbb{T})$ is the critical Sobolev space for the well-posedness of this model.

As noticed in [7], all suitable assumptions can always be imposed on the external source term f . Consequently, with only a minor loss of generality, one may assume

$$f = 0,$$

in order to carry out a qualitative study of the properties of this model, which are mainly governed by its linear and nonlinear terms. One such property concerns the global-in-time existence of solutions.

In full generality, smooth solutions formally satisfy the following energy estimate:

$$\frac{d}{dt} \|u(t, \cdot)\|_{L^2}^2 \lesssim - \int_{\mathbb{T}} (\mathcal{H}u) \partial_x (\mathcal{H}u) u \, dx - \|u(t, \cdot)\|_{L^2}^2,$$

where the nonlinear effects of the nonlocal term $(\mathcal{H}u) \partial_x (\mathcal{H}u)$ ultimately prevent the derivation of suitable energy estimates that would yield global-in-time existence of solutions through standard Grönwall-type arguments.

In this context, for $s \geq 0$, it is shown in [7] that small initial data $u_0 \in H_0^s(\mathbb{T})$ satisfying $\|u_0\|_{L^2} \leq \frac{\mu}{2}$, yield global-in-time solutions $u \in \mathcal{C}([0, +\infty[, H_0^s(\mathbb{T})))$ that satisfy the uniform bound $\|u(t, \cdot)\|_{L^2} \leq \frac{\mu}{4}$, for all $t > 0$.

The long-time dynamics of solutions arising from large initial data, *i.e.*, when $\|u_0\|_{L^2} > \frac{\mu}{2}$, appear to be considerably more complex. In fact, by exploiting the Fourier mode decomposition, the last sections of [7]

show, on the one hand, that certain large initial datum supported on a finite number of Fourier modes yield a global-in-time solution $u(t, x)$ to the model. On the other hand, this same type of initial data can lead to blow-up in finite time of the corresponding solution.

New theoretical results. The main objective of this note is to continue the qualitative study of the model introduced above. Specifically, we investigate the spatial asymptotic behavior of its solutions, assuming from now on that the spatial variable x belongs to the entire real line $\mathbb{R}$.

The spatial asymptotics of solutions to models in fluid motion are not only of mathematical interest but also of physical relevance. Intuitively, as mentioned in [4], if at the beginning of the evolution problem the fluid is at rest outside a bounded region (*i.e.*, the initial data are compactly supported), we would like to determine the rate at which the fluid particles move away from the initial region once the motion begins.

Mathematically, such questions are addressed by studying the *pointwise behavior* of solutions to these models as $|x| \rightarrow +\infty$. For instance, in the case of the *three-dimensional Navier–Stokes* equations on the whole space, it was shown in [12] that well-prepared initial data lead to an instantaneous spreading of the corresponding solutions, which cannot decay at infinity faster than the rate $1/|x|^4$. See also [2, 3, 4, 6] for further references on the spatial pointwise behavior of the Navier–Stokes equations.

Concerning some one-dimensional models, this question has been studied in [8, 9] for certain *dissipative modifications* of the *Korteweg–De Vries* and *Benjamin–Ono* equations, which arise in the study of viscous stratified fluids. In these cases, it was shown that well-prepared initial data lead to solutions with an optimal pointwise decay rate of $1/|x|^2$ as $|x| \rightarrow +\infty$. Recently, in [10], this study is done for a *generalized dispersive-dissipative Kuramoto-type equation*, leading pointwise decaying rates of solutions which are essentially governing by the fractional power of the Laplacian operator appearing in this equation.

Motivated by the aforementioned studies, we now turn our attention to the initial value problem associated with the nonlocal nonlinear model introduced above

$$(1) \quad \begin{cases} \partial_t u + u \partial_x u + (\mathcal{H}u) \partial_x (\mathcal{H}u) + \beta \mathcal{H}u - \mu \partial_x^2 u = 0, & \beta \neq 0, \quad \mu > 0, \\ u(0, \cdot) = u_0, \end{cases}$$

where, for a time $T > 0$, the function $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ denotes the solution, and $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ an initial datum. Additionally, the operator $\mathcal{H}$ is the Hilbert's transform, which can be defined in the Fourier level by the symbol

$$(2) \quad \widehat{\mathcal{H}\varphi}(\xi) = -i \operatorname{sign}(\xi) \widehat{\varphi}(\xi), \quad \operatorname{sign}(\xi) = \begin{cases} -1, & \xi < 0, \\ 0, & \xi = 0, \\ 1, & \xi > 0, \end{cases} \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

Note that, with a minor loss of generality, the external source term is taken to be zero. Here, we aim to understand how the linear and nonlinear terms in this equation govern the asymptotic spatial behavior of the solutions.

We will consider the following hypothesis on the initial data:

$$(3) \quad u_0 \in H^s(\mathbb{R}), \quad s > \frac{3}{2}.$$

This constraint on the regularity parameter s is essentially technical, introduced to handle the nonlinear terms $u \partial_x u$ and $(\mathcal{H}u) \partial_x (\mathcal{H}u)$ in equation (1).

Furthermore, for a constant $C_0 > 0$ and a fixed parameter $\gamma > 0$, we assume the prescribed pointwise decaying rate

$$(4) \quad |u_0(x)| \leq \frac{C_0}{(1 + |x|)^{1+\gamma}}, \quad \text{for any } x \in \mathbb{R}.$$

Here, the parameter γ essentially measures the decay rate assumed for the initial data. Nevertheless, due to certain technical constraints, a faster decay rate, given by the exponent $1 + \gamma$, is required.

Then, our main result is stated as follows:

Theorem 1.1. *Assume (3) and (4). There exists a time $T_0 = T_0(u_0) > 0$ and a unique solution $u \in \mathcal{C}([0, T_0], H^s(\mathbb{R}))$ of equation (1). Additionally, for any time $0 < t \leq T_0$ and $x \in \mathbb{R}$, the arising solution $u(t, x)$ of equation (1) satisfies the pointwise estimates:*

$$(5) \quad |u(t, x)| + |\mathcal{H}u(t, x)| \leq \frac{C_1}{t^{\frac{1}{2}}(1 + |x|)^{\min(1, \gamma)}},$$

where the constant $C_1 = C_1(\beta, \gamma, \mu, T_0, C_0, u) > 0$ is independent of the time variable t and the spatial variable x .

Estimate (5) exhibits a pointwise spatial decay rate for the solution $u(t, x)$, representing the horizontal velocity, and for its Hilbert transform $\mathcal{H}u(t, x)$, representing the vertical velocity in the context of the equatorial oceanic flows introduced above.

When comparing estimates (4) and (5), it is interesting to observe an instantaneous loss of persistence with respect to the prescribed decay rate assumed for the initial data. Specifically, as $|x| \rightarrow +\infty$ we have

$$|u_0(x)| \lesssim \frac{1}{|x|^{1+\gamma}} \quad \text{at } t = 0, \quad \text{and} \quad |u(t, x)| \lesssim \begin{cases} \frac{1}{|x|^\gamma}, & 0 < \gamma < 1, \\ \frac{1}{|x|}, & \gamma > 1, \end{cases} \quad \text{for } 0 < t \leq T_0.$$

We thus conclude that for small values of γ (that is, when $0 < \gamma < 1$) the decay rate of the initial data still influences the spatial asymptotics of the solution, whereas for large values of γ (when $\gamma > 1$), the solution decays at infinity like $1/|x|$.

In order to briefly explain this phenomenon, we mention that, roughly speaking, estimate (5) is obtained from the mild formulation of the solution $u(t, x)$, which involves a convolution kernel $K(t, x)$. This kernel, explicitly given in expression (11) below, depends strongly on the Coriolis parameter β . In the physically relevant case $\beta \neq 0$, we show that $K(t, x) \sim 1/|x|$ for any $t > 0$ and sufficiently large $|x|$, which leads to the observed decay rate of the solutions when $\gamma > 1$.

It is therefore interesting to observe the influence of the Coriolis parameter β on the spatial asymptotics of solutions to equation (1). In the case $\beta = 0$, expression (11) shows that $K(t, x)$ is the well-known heat kernel, which decays rapidly in the spatial variable. Nevertheless, this case is not physically relevant in the context of this equation.

Assuming fast-decaying data, which satisfy (4) for $\gamma > 1$, in our next result we shed some light on the optimality of the decay rate $1/|x|$ of solutions.

Proposition 1.1. *Under the same hypotheses (3) and (4), assume that $\gamma > 1$ and that the initial datum also satisfies*

$$(6) \quad \int_{\mathbb{R}} u_0(x) dx := M(u_0) \neq 0.$$

Let $u(t, x)$ be the associated solution of equation (1) obtained in Theorem 1.1. Then, one of the following statements does not hold:

- There exist $\varepsilon > 0$ and $M_\varepsilon > 0$ such that, for any $0 < t \leq T_0$ and any $|x| > M_\varepsilon$, the solution satisfies the pointwise decay estimate:

$$(7) \quad |u(t, x)| + |\mathcal{H}u(t, x)| \leq \frac{C_2}{t^{\frac{1}{2}} |x|^{1+\varepsilon}},$$

for some generic constant $C_2 > 0$, independent of t and x , but possibly depending on the parameters $\beta, \gamma, \varepsilon, \mu, T_0, C_0, M(u_0)$, as well as on certain norms of the solution u .

- The solution satisfies the mean decay property

$$(8) \quad u, \mathcal{H}u \in \mathcal{C}\left([0, T_0], L^2(\mathbb{R}, |x| dx)\right).$$

In order to explain in what sense this result suggests the optimality of the decay $1/|x|$, we provide a brief explanation of the main idea of the proof. First, we discuss the meaning of (7) and (8). Note that estimate (7) states that the solution $u(t, x)$ of equation (1) decays at infinity faster than $1/|x|$, while equation (8) shows that, for any time $0 \leq t \leq T_0$, the solution $u(t, x)$ satisfies $|x|u(t, \cdot) \in L^2(\mathbb{R})$. Such weighted Lebesgue spaces have been used in previous works concerning spatial decay in one-dimensional dispersive models [13, 14, 15]. Specifically, the fact that $|x|u(t, x)$ is an L^2 -function implies that, on average, $u(t, x)$ must decay at infinity faster than $1/|x|$.

In this context, assuming that both (7) and (8) hold, a contradiction arises. Indeed, under these assumptions, and assuming in addition the technical nonzero-mean condition on the initial data (6), we are able to show that, for any fixed time $0 < t \leq T_0$, the solution exhibits the following sharp asymptotic behavior:

$$u(t, x) = \frac{1}{x} \Phi(M(u_0), t, u) + o(t) \left(\frac{1}{|x|} \right), \quad |x| \rightarrow +\infty,$$

where the function $\Phi(M(u_0), t, u)$, explicitly defined in equation (54) below, does not depend on the spatial variable x , and the expression $o(t)(1/|x|)$ satisfies

$$\lim_{|x| \rightarrow +\infty} \frac{o(t)(1/|x|)}{1/|x|} = 0.$$

Ultimately, this asymptotic profile yields a contradiction with assumptions (7) and (8), respectively.

Returning to Theorem 1, note that, as a by-product, we also prove the local well-posedness of equation (1) in the space $H^s(\mathbb{R})$ for $s > \frac{3}{2}$. In this sense, we partially extend to the whole real line $\mathbb{R}$ the result obtained in [7, Proposition 3.3] for the torus $\mathbb{T}$. Additionally, following similar arguments as in [7, Proposition 3.4], the local well-posedness in H^s -spaces can be extended to $s > -\frac{1}{2}$. Moreover, the question of local well-posedness in the critical space $H^{-\frac{1}{2}}$, in both the periodic and non-periodic cases, appears to be far from obvious.

On the other hand, the global-in-time existence of H^s -solutions to equation (1) is an interesting question, since the arguments used in the periodic setting are no longer valid on the whole real line $\mathbb{R}$. In fact, the existence of small L^2 -solutions belonging to the space $\mathcal{C}([0, +\infty), H_0^s(\mathbb{T}))$ for $s \geq 0$, proved in [7, Proposition 4.1], relies crucially on Poincaré's inequality for zero-mean functions. Moreover, the finite-time blow-up of solutions for certain well-prepared large initial data, established in [7, Proposition 6.1], also exploits tools specific to the periodic setting, in particular, the Fourier-mode decomposition of periodic, zero-mean, square-integrable functions

Consequently, on the whole real line $\mathbb{R}$, the long-time dynamics of H^s -solutions to equation (1) remains, to the best of our knowledge, a challenging open problem. In future work, we aim to gain a deeper understanding of this issue. Nevertheless, by performing suitable energy estimates, we are able to establish the following blow-up criterion.

Proposition 1.2. *Under the same hypothesis as in (3), let $u \in \mathcal{C}([0, T_0], H^s(\mathbb{R}))$ be the solution to equation (1) obtained in Theorem 1.1. Then the following statement holds: for some time $T_0 < T_* < +\infty$, we have*

$$(9) \quad \lim_{t \rightarrow T_*} \|u(t, \cdot)\|_{H^s} = +\infty \quad \text{if and only if} \quad \int_0^{T_*} \|\partial_x u(t, \cdot)\|_{L^\infty} dt = +\infty.$$

From this blow-up criterion and well-known properties of the Hilbert transform, for a given time $T_0 < T_*$, assuming that

$$\int_0^{T_*} \|\partial_x u(t, \cdot)\|_{L^\infty} dt < +\infty,$$

it follows that the solution $u(t, x)$ of equation (1) satisfies

$$u, \mathcal{H}u \in \mathcal{C}([0, T_*], H^s(\mathbb{R})).$$

It is then natural to ask whether this information on u allows the pointwise spatial decay estimate (5) to be extended to all times $0 < t \leq T_*$. Nevertheless, to the best of our knowledge, this does not seem to be possible in full generality. In this setting, in order to extend the pointwise estimate (5) to further times, we require an additional technical assumption on the solution $u(t, x)$.

Proposition 1.3. *The pointwise estimates (5) for the solution $u(t, x)$ remain valid for all times $0 < t < T_*$, with $T_0 < T_* < +\infty$, provided that*

$$(10) \quad u, \mathcal{H}u \in \mathcal{C}([0, T_*], H^s \cap L^1(\mathbb{R})).$$

Even when the initial data belong to $L^1(\mathbb{R})$, it is not well understood whether this property is preserved by the associated solution of equation (1); see Remark 2 for more details on this point. Consequently, in this result we provide a partial answer to the question raised above.

Organization of the article. The rest of the paper is organized as follows. Our proof relies strongly on the mild formulation of equation (36), specifically on the convolution kernel arising from the linear terms. In Section 2, we introduce this kernel and derive several useful estimates. Section 3 is devoted to the proof of Theorem 1.1 and Proposition 1.1. Finally, in Section 4, we provide the proof of Propositions 1.2 and 1.3.

Notation. Throughout this note, $\widehat{\varphi}(\xi)$ denotes the Fourier transform of φ . Additionally, $C > 0$ denotes a generic constant, depending on the parameters of the model, which may change from one line to the next.

2. KERNEL ESTIMATES

We introduce the kernel $K(t, x)$ as the solution of the linear problem associated to equation (1):

$$\begin{cases} \partial_t K + \beta \mathcal{H}K - \mu \partial_x^2 K = 0, & \beta \neq 0, \quad \mu > 0, \\ K(t, 0) = \delta_0, \end{cases}$$

where δ_0 denotes the Dirac mass at the origin. A direct computation yields the explicit expression for $K(t, x)$:

$$(11) \quad K(t, x) = \int_{-\infty}^{+\infty} e^{2\pi i x \xi} e^{-\mu \xi^2 t + i\beta \text{sign}(\xi)t} d\xi.$$

Using this representation, and following some ideas in [8, 9, 10], we derive some useful pointwise estimates for $K(t, x)$ and $\mathcal{H}K(t, x)$.

Proposition 2.1. *There exists a constant $C = C(\beta, \mu) > 0$, which depends on the parameters $\beta \neq 0$ and $\mu > 0$, such that for any $t > 0$ and $x \in \mathbb{R}$ it holds:*

$$(12) \quad |K(t, x)| + |\mathcal{H}K(t, x)| \leq C \frac{\eta(t)}{t^{\frac{1}{2}}} \frac{1}{1 + |x|}, \quad \eta(t) := 1 + t^{\frac{1}{2}} + t,$$

where $\mathcal{H}$ is the Hilbert's transform defined in expression (2).

Proof. We begin by considering the function $K(t, x)$. Using expression (11), the identity $\partial_\xi e^{2\pi i x \xi} = 2\pi i x e^{2\pi i x \xi}$, and integrating by parts, we obtain

$$\begin{aligned}
 K(t, x) &= \int_{-\infty}^0 e^{2\pi i x \xi} e^{-\mu \xi^2 t - i\beta t} d\xi + \int_0^{+\infty} e^{2\pi i x \xi} e^{-\mu \xi^2 t + i\beta t} d\xi \\
 &= \frac{1}{2\pi i x} \int_{-\infty}^0 \partial_\xi (e^{2\pi i x \xi}) e^{-\mu \xi^2 t - i\beta t} d\xi + \frac{1}{2\pi i x} \int_0^{+\infty} \partial_\xi (e^{2\pi i x \xi}) e^{-\mu \xi^2 t + i\beta t} d\xi \\
 &= -\frac{1}{2\pi i x} \int_{-\infty}^0 e^{2\pi i x \xi} \partial_\xi (e^{-\mu \xi^2 t - i\beta t}) d\xi + \frac{1}{2\pi i x} \left(e^{2\pi i x \xi} e^{-\mu \xi^2 t - i\beta t} \Big|_{-\infty}^0 \right) \\
 &\quad - \frac{1}{2\pi i x} \int_0^{+\infty} e^{2\pi i x \xi} \partial_\xi (e^{-\mu \xi^2 t + i\beta t}) d\xi + \frac{1}{2\pi i x} \left(e^{2\pi i x \xi} e^{-\mu \xi^2 t + i\beta t} \Big|_0^{+\infty} \right) \\
 &= \frac{e^{-i\beta t} - e^{i\beta t}}{2\pi i x} - \underbrace{\frac{1}{2\pi i x} \left(\int_{-\infty}^0 e^{2\pi i x \xi} \partial_\xi (e^{-\mu \xi^2 t - i\beta t}) d\xi + \int_0^{+\infty} e^{2\pi i x \xi} \partial_\xi (e^{-\mu \xi^2 t + i\beta t}) d\xi \right)}_{I(t, x)}.
 \end{aligned} \tag{13}$$

Here, for any $x \neq 0$, by direct calculation we obtain

$$\frac{e^{-i\beta t} - e^{i\beta t}}{2\pi i x} = -\frac{\sin(\beta t)}{\pi x}.$$

Additionally, for any $x \neq 0$, the second term is estimated as follows:

$$I(t, x) \leq C \frac{t^{\frac{1}{2}}}{x^2}. \tag{14}$$

In fact, following similar computations as above and using the rapid decay properties of the heat kernel $e^{-\mu \xi^2 t}$, we write

$$\begin{aligned}
 I(t, x) &= \frac{1}{(2\pi i x)^2} \left(-\int_{-\infty}^0 e^{2\pi i x \xi} \partial_\xi^2 (e^{-\mu \xi^2 t - i\beta t}) d\xi + e^{2\pi i x \xi} e^{-\mu \xi^2 t - i\beta t} (-2\mu \xi t) \Big|_{-\infty}^0 \right. \\
 &\quad \left. - \int_0^{+\infty} e^{2\pi i x \xi} \partial_\xi^2 (e^{-\mu \xi^2 t + i\beta t}) d\xi + e^{2\pi i x \xi} e^{-\mu \xi^2 t + i\beta t} (-2\mu \xi t) \Big|_0^{+\infty} \right) \\
 &= \frac{1}{4\pi^2 x^2} \left(\int_{-\infty}^0 e^{2\pi i x \xi} e^{-\mu \xi^2 t - i\beta t} (4\mu^2 \xi^2 t^2 - 2\mu t) d\xi + \int_0^{+\infty} e^{2\pi i x \xi} e^{-\mu \xi^2 t + i\beta t} (4\mu^2 \xi^2 t^2 - 2\mu t) d\xi \right) \\
 &\leq \frac{1}{4\pi^2 x^2} \int_{-\infty}^{+\infty} e^{-\mu \xi^2 t} (4\mu^2 \xi^2 t^2 + 2\mu t) d\xi = \frac{t}{4\pi^2 x^2} \int_{-\infty}^{+\infty} e^{-\mu(t^{\frac{1}{2}} \xi)^2} (4\mu^2 (t^{\frac{1}{2}} \xi)^2 + 2\mu) d\xi \leq C \frac{t^{\frac{1}{2}}}{x^2}.
 \end{aligned}$$

Once we have established these identities and estimates, returning to (13), we obtain for $|x| > 1$

$$|K(t, x)| \leq C \frac{|\sin(\beta t)|}{|x|} + C \frac{t^{\frac{1}{2}}}{|x|^2} \leq C (1 + t^{\frac{1}{2}}) \frac{1}{|x|}. \tag{15}$$

Moreover, we deduce the following asymptotic profile, which will be used later:

$$K(t, x) = -\frac{\sin(\beta t)}{x} + I(t, x), \quad t > 0, \quad |x| > 1. \tag{16}$$

On the other hand, for any $x \in \mathbb{R}$ we can write

$$|K(t, x)| \leq \|K(t, \cdot)\|_{L^\infty} \leq \|\widehat{K}(t, \cdot)\|_{L^1} \leq \int_{-\infty}^{+\infty} e^{-\mu \xi^2 t} d\xi \leq \frac{C}{t^{\frac{1}{2}}}. \tag{17}$$

Therefore, by combining estimates (15) and (17), we arrive at the first desired estimate (12).

The estimate concerning $\mathcal{H}K(t, x)$ follows from similar computations. For the reader's convenience, we present below the key calculation.

$$\begin{aligned} \mathcal{H}K(t, x) &= -i \int_{-\infty}^{+\infty} e^{2\pi i x \xi} \operatorname{sign}(\xi) e^{-\mu \xi^2 t + i \operatorname{sign}(\xi) \beta t} d\xi \\ &= \frac{i}{2\pi i x} \int_{-\infty}^0 \partial_\xi (e^{2\pi i x \xi}) e^{-\mu \xi^2 t - i \beta t} d\xi - \frac{i}{2\pi i x} \int_0^{+\infty} \partial_\xi (e^{2\pi i x \xi}) e^{-\mu \xi^2 t + i \beta t} d\xi \\ &= \frac{e^{-i\beta t} + e^{i\beta t}}{2\pi i x} - \underbrace{\frac{1}{2\pi i x} \left(\int_{-\infty}^0 e^{2\pi i x \xi} \partial_\xi (e^{-\mu \xi^2 t - i \beta t}) d\xi - \int_0^{+\infty} e^{2\pi i x \xi} \partial_\xi (e^{-\mu \xi^2 t + i \beta t}) d\xi \right)}_{J(t, x)}, \end{aligned}$$

As above, for $x \neq 0$ we have

$$\frac{e^{-i\beta t} + e^{i\beta t}}{2\pi i x} = \frac{\cos(\beta t)}{\pi i x}, \quad \text{and} \quad J(t, x) \leq C \frac{t^{\frac{1}{2}}}{x^2},$$

which yields the asymptotic profile

$$(18) \quad \mathcal{H}K(t, x) = \frac{\cos(\beta t)}{\pi i x} + J(t, x), \quad t > 0, \quad |x| > 1.$$

This completes the proof of Proposition 2.1. $\square$

Remark 1. Note that the asymptotic profile (16) shows that, for $\beta \neq 0$ and any time $t \neq \frac{k\pi}{\beta}$ with $k \in \mathbb{Z}$, the kernel $K(t, x)$ has the optimal decay rate $1/|x|$. Therefore, $K(t, \cdot) \notin L^1(\mathbb{R})$. Similarly, from the asymptotic profile (18), the same conclusion holds for $\mathcal{H}K(t, x)$ for any time $t \neq \frac{(2k+1)\pi}{2\beta}$.

Consequently, when $\beta \neq 0$, there is no positive time t for which both $K(t, x)$ and $\mathcal{H}K(t, x)$ decay at infinity faster than $1/|x|$, and for which $K(t, \cdot)$ and $\mathcal{H}K(t, \cdot)$ belong to $L^1(\mathbb{R})$.

3. ASYMPTOTIC BEHAVIOR IN THE SPATIAL VARIABLE

3.1. Proof of Theorem 1.1. We begin by outlining the general idea of the proof. To our knowledge, the Hilbert transform $\mathcal{H}$ is not bounded on the weighted space $L^\infty((1 + |\cdot|)^{\min(1, \gamma)} dx)$, which makes it difficult to control the nonlinear term $(\mathcal{H}u) \partial_x (\mathcal{H}u)$.

To overcome this issue, in the first step of the proof we introduce an auxiliary coupled system in the variables $\tilde{u}$ and $\tilde{v}$, which will allow to control both u and $\mathcal{H}u$ later. We then prove the existence of local-in-time $\mathcal{C}_t H_x^s$ -solutions, which also satisfy

$$\left\| (1 + |x|)^{\min(1, \gamma)} (\tilde{u}, \tilde{v})(t, \cdot) \right\|_{L^\infty} < +\infty.$$

In the second step, we prove the uniqueness of solutions $(\tilde{u}, \tilde{v})$ to this auxiliary system in the larger space $\mathcal{C}_t H_x^s$.

Finally, in the third step, returning to the original equation (1) we obtain a solution $u \in \mathcal{C}_t H_x^s$, where the pair $(u, \mathcal{H}u)$ also satisfies the auxiliary system introduced above. By uniqueness, it follows that $(u, \mathcal{H}u) = (\tilde{u}, \tilde{v})$, which leads to the pointwise estimate (5).

First step: the auxiliary system. In equation (1), observe that if $\tilde{u}(t, x)$ is a solution with initial datum $\tilde{u}_0$, then $\tilde{v}(t, x) := \mathcal{H}\tilde{u}(t, x)$ formally satisfies:

$$\partial_t \tilde{v} + \mathcal{H}(\tilde{u} \partial_x \tilde{u}) + \mathcal{H}(\tilde{v} \partial_x \tilde{v}) + \beta \mathcal{H}\tilde{v} - \mu \partial_x^2 \tilde{v} = 0, \quad \tilde{v}(0, \cdot) = \mathcal{H}\tilde{u}_0.$$

Therefore, we will study both $\tilde{u}(t, x)$ and $\tilde{v}(t, x)$ as solutions of the following coupled system:

$$(19) \quad \begin{cases} \partial_t \tilde{u} + \tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v} + \beta \mathcal{H} \tilde{u} - \mu \partial_x^2 \tilde{u} = 0, \\ \partial_t \tilde{v} + \mathcal{H}(\tilde{u} \partial_x \tilde{u}) + \mathcal{H}(\tilde{v} \partial_x \tilde{v}) + \beta \mathcal{H} \tilde{v} - \mu \partial_x^2 \tilde{v} = 0, \\ \tilde{u}(0, \cdot) = \tilde{u}_0, \quad \tilde{v}(0, \cdot) = \mathcal{H} \tilde{u}_0. \end{cases}$$

Let $T > 0$ a time to be fixed later. For $s > \frac{3}{2}$, consider the Banach space

$$E_T = \{w \in \mathcal{C}([0, T], H^s(\mathbb{R})) : \|w\|_{E_T} < +\infty\},$$

endowed with the norm

$$\|w\|_{E_T} := \sup_{0 \leq t \leq T} \|w(t, \cdot)\|_{H^s} + \sup_{0 < t \leq T} t^{\frac{1}{2}} \left\| (1 + |x|)^{\min(1, \gamma)} w(t, \cdot) \right\|_{L^\infty}.$$

Here, the second term characterizes the pointwise decay in the spatial variable of the function $w(t, x)$. The time weight $t^{\frac{1}{2}}$ is included for technical reasons to ensure the validity of our subsequent estimates.

Proposition 3.1. *Assume that $\tilde{u}_0$ satisfies (3) and (4). There exists a time $\tilde{T}_0 = \tilde{T}_0(\tilde{u}_0) > 0$ and a solution $(\tilde{u}, \tilde{v}) \in E_{\tilde{T}_0}$ to the coupled system (19). In particular, for any $0 < t \leq \tilde{T}_0$ and $x \in \mathbb{R}$, this solution verifies the pointwise estimate:*

$$(20) \quad |\tilde{u}(t, x)| + |\tilde{v}(t, x)| \leq \frac{\|\tilde{u}\|_{E_{\tilde{T}_0}} + \|\tilde{v}\|_{E_{\tilde{T}_0}}}{t^{\frac{1}{2}} (1 + |x|)^{\min(1, \gamma)}}.$$

Proof. We observe that system (19) can be equivalently reformulated as the following fixed-point problem:

$$(21) \quad \begin{cases} \tilde{u}(t, \cdot) = K(t, \cdot) * \tilde{u}_0 - \int_0^t K(t - \tau, \cdot) * (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, \cdot) d\tau, \\ \tilde{v}(t, \cdot) = \mathcal{H}K(t, \cdot) * \tilde{u}_0 - \int_0^t \mathcal{H}K(t - \tau, \cdot) * (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, \cdot) d\tau. \end{cases}$$

In the second equation, the Hilbert transform is applied to the kernel $K(t, \cdot)$, defined in expression (11). We will therefore solve this system in the Banach space E_T introduced above.

In the following technical lemmas, we study the linear and nonlinear terms separately. For the linear term involving the initial data, note that by (4), since $\gamma > 0$, we have $\tilde{u}_0 \in L^1(\mathbb{R})$.

Lemma 3.1. *Under assumptions (3) and (4), the following estimate holds:*

$$(22) \quad \|K(t, \cdot) * \tilde{u}_0\|_{E_T} + \|\mathcal{H}K(t, \cdot) * \tilde{u}_0\|_{E_T} \leq C \eta(T) (\|\tilde{u}_0\|_{H^s} + \|\tilde{u}_0\|_{L^1} + C_0),$$

where $C > 0$ is a constant depending on the parameters β, γ, μ , and $\eta(T) > 0$ is given in (12).

Proof. From (11), for any $t > 0$ and $\xi \in \mathbb{R}$ we have $|\hat{K}(t, \xi)| \leq 1$. Consequently, the first term in the norm $\|\cdot\|_{E_T}$ is directly estimated as:

$$(23) \quad \sup_{0 \leq t \leq T} (\|K(t, \cdot) * \tilde{u}_0\|_{H^s} + \|\mathcal{H}K(t, \cdot) * \tilde{u}_0\|_{H^s}) \leq C \|\tilde{u}_0\|_{H^s}.$$

We focus on the second term in $\|\cdot\|_{E_T}$. For the expression $K(t, \cdot) * \tilde{u}_0$, for any fixed $x \neq 0$, we write

$$|K(t, \cdot) * \tilde{u}_0(x)| \leq \int_{\mathbb{R}} |K(t, x - y)| |\tilde{u}_0(y)| dy = \underbrace{\int_{|y| \leq \frac{|x|}{2}} |K(t, x - y)| |\tilde{u}_0(y)| dy}_{I_1(t, x)} + \underbrace{\int_{|y| > \frac{|x|}{2}} |K(t, x - y)| |\tilde{u}_0(y)| dy}_{I_2(t, x)}.$$

For the term $I_1(t, x)$, since $|y| \leq \frac{|x|}{2}$, it follows that $|x - y| \geq |x| - |y| \geq \frac{|x|}{2}$. Then, using the estimate (12), and since $0 < t \leq T$, we obtain

$$(24) \quad |K(t, x - y)| \leq C \frac{\eta(t)}{t^{\frac{1}{2}}} \frac{1}{|x - y|} \leq C \frac{\eta(T)}{t^{\frac{1}{2}}} \frac{1}{1 + |x|}.$$

Additionally, recalling that $\tilde{u}_0 \in L^1(\mathbb{R})$, we have

$$(25) \quad I_1(t, x) \leq C \frac{\eta(T)}{t^{\frac{1}{2}}} \frac{1}{1 + |x|} \int_{|y| \leq \frac{|x|}{2}} |\tilde{u}_0(y)| dy \leq C \frac{\eta(T)}{t^{\frac{1}{2}}} \frac{1}{1 + |x|} \|\tilde{u}_0\|_{L^1} \leq C \frac{\eta(T)}{t^{\frac{1}{2}}} \frac{1}{(1 + |x|)^{\min(1, \gamma)}} \|\tilde{u}_0\|_{L^1}.$$

For the term $I_2(t, x)$, since $|y| > \frac{|x|}{2}$, by assumption (4) we have

$$|\tilde{u}_0(y)| \leq \frac{C_0}{(1 + |y|)^{1 + \gamma}} = \frac{C_0}{(1 + |y|)(1 + |y|)^\gamma} \leq \frac{C_0}{(1 + |x|)^\gamma} \frac{C}{1 + |y|}.$$

Fixed a parameter $1 < q < +\infty$. Then $\left\| \frac{1}{1 + |y|} \right\|_{L^q} \leq C_q < +\infty$. Choose $1 < p < +\infty$ such that $1 = 1/p + 1/q$. By (12), for $0 < t \leq T$, we obtain

$$(26) \quad \|K(t, \cdot)\|_{L^p} \leq C \frac{\eta(T)}{t^{\frac{1}{2}}} \left\| \frac{1}{1 + |x|} \right\|_{L^p} \leq C_p \frac{\eta(T)}{t^{\frac{1}{2}}}.$$

Thus, applying the Hölder inequality, we deduce

$$(27) \quad \begin{aligned} I_2(t, x) &\leq \frac{C_0}{(1 + |x|)^\gamma} \int_{|y| > \frac{|x|}{2}} |K(t, x - y)| \frac{1}{1 + |y|} dy \leq \frac{C_0}{(1 + |x|)^\gamma} \|K(t, \cdot)\|_{L^p} \left\| \frac{1}{1 + |y|} \right\|_{L^q} \\ &\leq \frac{C_0}{(1 + |x|)^\gamma} \frac{C_{p,q} \eta(T)}{t^{\frac{1}{2}}} \leq \frac{C_0}{(1 + |x|)^{\min(1, \gamma)}} \frac{C \eta(T)}{t^{\frac{1}{2}}}. \end{aligned}$$

Gathering estimates (25) and (27), we obtain

$$(28) \quad \sup_{0 \leq t \leq T} t^{\frac{1}{2}} \left\| (1 + |x|)^{\min(1, \gamma)} K(t, \cdot) * \tilde{u}_0 \right\|_{L^\infty} \leq C \eta(T) (\|\tilde{u}_0\|_{L^1} + C_0).$$

Finally, the expression $\mathcal{H}K(t, \cdot) * \tilde{u}_0$, follows from the same arguments, using the pointwise estimate for $\mathcal{H}K(t, x)$ given in (12). $\square$

Lemma 3.2. *The following estimates hold:*

$$(29) \quad \begin{aligned} &\left\| \int_0^t K(t - \tau, \cdot) * (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, \cdot) d\tau \right\|_{E_T} + \left\| \int_0^t \mathcal{H}K(t - \tau, \cdot) * (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, \cdot) d\tau \right\|_{E_T} \\ &\leq C T^{\frac{1}{2}} \left(1 + \eta(T) \left(1 + T^{\frac{1}{2}} \right) \right) (\|\tilde{u}\|_{E_T} + \|\tilde{v}\|_{E_T})^2. \end{aligned}$$

Proof. As before, it is enough to consider the expression $\int_0^t K(t - \tau, \cdot) * (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, \cdot) d\tau$. Using known estimates for $\mathcal{H}K(t - \tau, \cdot)$, the expression $\int_0^t \mathcal{H}K(t - \tau, \cdot) * (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, \cdot) d\tau$ follows the same arguments.

We start by studying the first term in the norm $\|\cdot\|_{E_T}$, to this end, we will write $\tilde{u} \partial_x \tilde{u}$ as $\frac{1}{2} \partial_x (\tilde{u}^2)$. First, observe that by (11), for any $t > 0$ we have $\left\| |\xi| \widehat{K}(t, \cdot) \right\|_{L^\infty} \leq \frac{C}{t^{\frac{1}{2}}}$. Additionally, using well-known properties

of the Hilbert transform $\mathcal{H}$, along with the fact that $H^s(\mathbb{R})$ is an Banach algebra for $s > \frac{3}{2}$, it follows that:

$$\begin{aligned}
 (30) \quad & \sup_{0 \leq t \leq T} \left\| \frac{1}{2} \int_0^t K(t-\tau, \cdot) * \partial_x (\tilde{u}^2 + \tilde{v}^2)(\tau, \cdot) d\tau \right\|_{H^s} \\
 & \leq \sup_{0 \leq t \leq T} \int_0^t \frac{C}{(t-\tau)^{\frac{1}{2}}} (\|\tilde{u}(\tau, \cdot)\|_{H^s}^2 + \|\tilde{v}(\tau, \cdot)\|_{H^s}^2) d\tau \\
 & \leq C T^{\frac{1}{2}} \sup_{0 \leq t \leq T} (\|\tilde{u}(t, \cdot)\|_{H^s}^2 + \|\tilde{v}(t, \cdot)\|_{H^s}^2) \\
 & \leq C T^{\frac{1}{2}} (\|\tilde{u}\|_{E_T} + \|\tilde{v}\|_{E_T})^2.
 \end{aligned}$$

Now, we will focus on the second term in the norm $\|\cdot\|_{E_T}$. For any fixed $x \neq 0$, we write

$$\begin{aligned}
 (31) \quad & \int_0^t K(t-\tau, \cdot) * (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, x) d\tau = \underbrace{\int_0^t \int_{|y| \leq \frac{|x|}{2}} K(t-\tau, x-y) (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, y) dy d\tau}_{I_1(t, x)} \\
 & + \underbrace{\int_0^t \int_{|y| > \frac{|x|}{2}} K(t-\tau, x-y) (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, y) dy d\tau}_{I_2(t, x)}.
 \end{aligned}$$

where we will estimate each term separately.

For the term $I_1(t, x)$, since $|y| \leq \frac{|x|}{2}$ and $0 < t-\tau < T$, by estimate (24) we have

$$|K(t-\tau, x-y)| \leq C \frac{\eta(T)}{(t-\tau)^{\frac{1}{2}}} \frac{1}{1+|x|}.$$

Then, using the Cauchy-Schwarz inequality, the continuous embeddings $H^s(\mathbb{R}) \subset H^1(\mathbb{R}) \subset L^2(\mathbb{R})$ and the fact that $0 < t \leq T$, we find

$$\begin{aligned}
 (32) \quad I_1(t, x) & \leq C \frac{\eta(T)}{1+|x|} \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \int_{|y| \leq \frac{|x|}{2}} (|\tilde{u}(\tau, y)| |\partial_y \tilde{u}(\tau, y)| + |\tilde{v}(\tau, y)| |\partial_y \tilde{v}(\tau, y)|) dy d\tau \\
 & \leq C \frac{\eta(T)}{1+|x|} \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \int_{\mathbb{R}} (|\tilde{u}(\tau, y)| |\partial_y \tilde{u}(\tau, y)| + |\tilde{v}(\tau, y)| |\partial_y \tilde{v}(\tau, y)|) dy d\tau \\
 & \leq C \frac{\eta(T)}{1+|x|} \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} (\|\tilde{u}(\tau, \cdot)\|_{L^2} \|\tilde{u}(\tau, \cdot)\|_{H^s} + \|\tilde{v}(\tau, \cdot)\|_{L^2} \|\tilde{v}(\tau, \cdot)\|_{H^s}) d\tau \\
 & \leq C \frac{\eta(T)}{1+|x|} \left(\int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \right) \sup_{0 \leq \tau \leq t} (\|\tilde{u}(\tau, \cdot)\|_{H^s}^2 + \|\tilde{v}(\tau, \cdot)\|_{H^s}^2) \\
 & \leq C \frac{\eta(T)}{1+|x|} T^{\frac{1}{2}} (\|\tilde{u}\|_{E_T}^2 + \|\tilde{v}\|_{E_T}^2) \\
 & \leq C \frac{\eta(T) T^{\frac{1}{2}}}{(1+|x|)^{\min(1, \gamma)}} (\|\tilde{u}\|_{E_T} + \|\tilde{v}\|_{E_T})^2.
 \end{aligned}$$

For the term $I_2(t, x)$, since $\tilde{u}, \tilde{v} \in E_T$ and $|y| > \frac{|x|}{2}$, we have

$$|\tilde{u}(\tau, y)| \leq \frac{1}{\tau^{\frac{1}{2}}} \frac{1}{(1+|y|)^{\min(1, \gamma)}} \|\tilde{u}\|_{E_T} \leq \frac{C}{\tau^{\frac{1}{2}}} \frac{1}{(1+|x|)^{\min(1, \gamma)}} \|\tilde{u}\|_{E_T},$$

and similarly

$$|\tilde{v}(\tau, y)| \leq \frac{C}{\tau^{\frac{1}{2}}} \frac{1}{(1+|x|)^{\min(1, \gamma)}} \|\tilde{v}\|_{E_T}.$$

Additionally, note that by the first estimate in (12), and since $0 < t - \tau < T$, it follows that

$$\|K(t - \tau, \cdot)\|_{L^2} \leq C \frac{\eta(T)}{(t - \tau)^{\frac{1}{2}}}.$$

Therefore, using again the Cauchy-Schwarz inequality and the fact that $\int_0^t \frac{d\tau}{\tau^{\frac{1}{2}}(t - \tau)^{\frac{1}{2}}} \leq C < +\infty$, we can write

$$\begin{aligned} I_2(t, x) &\leq C \frac{\eta(T)}{(1 + |x|)^{\min(1, \gamma)}} \|\tilde{u}\|_{E_T} \int_0^t \frac{1}{\tau^{\frac{1}{2}}} \int_{\mathbb{R}} |K(x - y, t - \tau)| |\partial_y \tilde{u}(\tau, y)| dy d\tau \\ &\quad + C \frac{\eta(T)}{(1 + |x|)^{\min(1, \gamma)}} \|\tilde{v}\|_{E_T} \int_0^t \frac{1}{\tau^{\frac{1}{2}}} \int_{\mathbb{R}} |K(x - y, t - \tau)| |\partial_y \tilde{v}(\tau, y)| dy d\tau \\ (33) \quad &\leq C \frac{\eta(T)}{(1 + |x|)^{\min(1, \gamma)}} \|\tilde{u}\|_{E_T} \int_0^t \frac{1}{\tau^{\frac{1}{2}}(t - \tau)^{\frac{1}{2}}} \|\tilde{u}(\tau, \cdot)\|_{H^1} d\tau \\ &\quad + C \frac{\eta(T)}{(1 + |x|)^{\min(1, \gamma)}} \|\tilde{v}\|_{E_T} \int_0^t \frac{1}{\tau^{\frac{1}{2}}(t - \tau)^{\frac{1}{2}}} \|\tilde{v}(\tau, \cdot)\|_{H^1} d\tau \\ &\leq C \frac{\eta(T)}{(1 + |x|)^{\min(1, \gamma)}} (\|\tilde{u}\|_{E_T} + \|\tilde{v}\|_{E_T})^2. \end{aligned}$$

Gathering estimates (32) and (33) into identity (31), it follows that

$$\begin{aligned} &\sup_{0 \leq t \leq T} t^{\frac{1}{2}} \left\| (1 + |x|)^{\min(1, \gamma)} \int_0^t K(t - \tau, \cdot) * (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, \cdot) d\tau \right\|_{L^\infty} \\ &\leq CT^{\frac{1}{2}} \eta(T) (1 + T^{\frac{1}{2}}) (\|\tilde{u}\|_{E_T} + \|\tilde{v}\|_{E_T})^2. \end{aligned}$$

□

Once estimates (22) and (29) have been established, we choose a time $\tilde{T}_0 \leq 1$. From the expression of $\eta(\tilde{T}_0)$ in (12), it follows that $\eta(\tilde{T}_0) \leq C$ and $1 + \eta(\tilde{T}_0) \left(1 + \tilde{T}_0^{\frac{1}{2}}\right) \leq C$, for some numerical constant $C > 0$. Consequently, estimates (22) and (29) take the form

$$\begin{aligned} &\|K(t, \cdot) * \tilde{u}_0\|_{E_{\tilde{T}_0}} + \|\mathcal{H}K(t, \cdot) * \tilde{u}_0\|_{E_{\tilde{T}_0}} \leq C (\|\tilde{u}_0\|_{H^s} + \|\tilde{u}_0\|_{L^1} + C_0), \\ &\left\| \int_0^t K(t - \tau, \cdot) * (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, \cdot) d\tau \right\|_{E_{\tilde{T}_0}} + \left\| \int_0^t \mathcal{H}K(t - \tau, \cdot) * (\tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_x \tilde{v})(\tau, \cdot) d\tau \right\|_{E_{\tilde{T}_0}} \\ &\leq C \tilde{T}_0^{\frac{1}{2}} (\|\tilde{u}\|_{E_T} + \|\tilde{v}\|_{E_T})^2. \end{aligned}$$

Next, we choose $\tilde{T}_0$ sufficiently small, namely

$$\tilde{T}_0 < \frac{1}{(4C(\|\tilde{u}_0\|_{H^s} + \|\tilde{u}_0\|_{L^1} + C_0))^2},$$

so that the Picard iteration scheme yields a solution $(\tilde{u}, \tilde{v}) \in E_{\tilde{T}_0}$ to system (21). This completes the proof of Proposition 3.1. □

Second step: uniqueness of solutions of the system (19).

Proposition 3.2. *For any fixed time $0 < T < +\infty$, the system (19) admits a unique solution in the space $\mathcal{C}([0, T], H^s(\mathbb{R}))$, with $s > \frac{3}{2}$.*

Proof. Suppose $(\tilde{u}_1, \tilde{v}_1), (\tilde{u}_2, \tilde{v}_2) \in \mathcal{C}([0, T], H^s(\mathbb{R}))$ are two solutions corresponding to the same initial datum $\tilde{u}_0$, and define $\mathbf{u} := \tilde{u}_1 - \tilde{u}_2$, $\mathbf{v} := \tilde{v}_1 - \tilde{v}_2$. Then $\mathbf{u}$ and $\mathbf{v}$ satisfy:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \partial_x \tilde{u}_1 + \tilde{u}_2 \partial_x \mathbf{u} + \mathbf{v} \partial_x \tilde{v}_1 + \tilde{v}_2 \partial_x \mathbf{v} + \beta \mathcal{H} \mathbf{u} - \mu \partial_x^2 \mathbf{u} = 0, \\ \partial_t \mathbf{v} + \mathcal{H}(\mathbf{u} \partial_x \tilde{u}_1 + \tilde{u}_2 \partial_x \mathbf{u} + \mathbf{v} \partial_x \tilde{v}_1 + \tilde{v}_2 \partial_x \mathbf{v}) + \beta \mathcal{H} \mathbf{v} - \mu \partial_x^2 \mathbf{v} = 0. \end{cases}$$

Hence, we obtain:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t, \cdot)\|_{L^2}^2 &= - \underbrace{\int_{\mathbb{R}} (\mathbf{u} \partial_x u_1) \mathbf{u} dx}_{(A_1)} - \underbrace{\int_{\mathbb{R}} (u_2 \partial_x \mathbf{u}) \mathbf{u} dx}_{(B_1)} - \underbrace{\int_{\mathbb{R}} (\mathbf{v} \partial_x v_1) \mathbf{u} dx}_{(A_2)} - \underbrace{\int_{\mathbb{R}} (v_2 \partial_x \mathbf{v}) \mathbf{u} dx}_{(B_2)} \\ &\quad - \mu \|\mathbf{u}(t, \cdot)\|_{H^1}^2, \\ \frac{1}{2} \frac{d}{dt} \|\mathbf{v}(t, \cdot)\|_{L^2}^2 &= - \underbrace{\int_{\mathbb{R}} \mathcal{H}(\mathbf{u} \partial_x u_1) \mathbf{v} dx}_{(A_3)} - \underbrace{\int_{\mathbb{R}} \mathcal{H}(u_2 \partial_x \mathbf{u}) \mathbf{v} dx}_{(B_3)} - \underbrace{\int_{\mathbb{R}} \mathcal{H}(\mathbf{v} \partial_x v_1) \mathbf{v} dx}_{(A_4)} - \underbrace{\int_{\mathbb{R}} \mathcal{H}(v_2 \partial_x \mathbf{v}) \mathbf{v} dx}_{(B_4)} \\ &\quad - \mu \|\mathbf{v}(t, \cdot)\|_{H^1}^2. \end{aligned}$$

Here, we need to estimate the mixed terms on the right-hand side. Some of them can be handled using similar arguments; therefore, we will only outline the main ideas.

The terms labeled (A_i) , for $i = 1, \dots, 4$, share the generic form $\int_{\mathbb{R}} (\mathbf{f} \partial_x h) \mathbf{g} dx$, with $\mathbf{f}, \mathbf{g} \in \{\mathbf{u}, \mathbf{v}, \mathcal{H}\mathbf{v}\}$ and $h \in \{\tilde{u}_1, \tilde{v}_1\}$. Using Hölder's inequality, standard properties of the Hilbert transform, and the Sobolev embedding $H^{s-1}(\mathbb{R}) \subset L^\infty(\mathbb{R})$ (since $s - 1 > \frac{1}{2}$), it follows that:

$$\left| \int_{\mathbb{R}} (\mathbf{f} \partial_x h) \mathbf{g} dx \right| \leq \|\mathbf{f} \partial_x h\|_{L^2} \|\mathbf{g}\|_{L^2} \leq \|\partial_x h\|_{L^\infty} \|\mathbf{f}\|_{L^2} \|\mathbf{g}\|_{L^2} \leq C \|h\|_{H^s} (\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2).$$

Thus, we obtain:

$$(34) \quad \sum_{i=1}^4 |(A_i)| \leq C (\|\tilde{u}_1\|_{H^s} + \|\tilde{v}_1\|_{H^s}) (\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{v}\|_{L^2}^2).$$

The terms labeled (B_i) have the generic expression $\int_{\mathbb{R}} (h \partial_x \mathbf{f}) \mathbf{g} dx$, where $\mathbf{f}$ and $\mathbf{g}$ are as above, and $h \in \{\tilde{u}_2, \tilde{v}_2\}$. Using arguments analogous to the previous ones, we then write:

$$\left| \int_{\mathbb{R}} (h \partial_x \mathbf{f}) \mathbf{g} dx \right| \leq \|h \partial_x \mathbf{f}\|_{L^2} \|\mathbf{g}\|_{L^2} \leq \|\partial_x \mathbf{f}\|_{L^2} \|h\|_{L^\infty} \|\mathbf{g}\|_{L^2} \leq C \|\mathbf{f}\|_{\dot{H}^1} \|h\|_{H^s} \|\mathbf{g}\|_{L^2}.$$

Applying the discrete Young inequalities, it follows that

$$C \|\mathbf{f}\|_{\dot{H}^1} \|h\|_{H^s} \|\mathbf{g}\|_{L^2} \leq \mu \|\mathbf{f}\|_{\dot{H}^1}^2 + \frac{C}{\mu} \|h\|_{H^s}^2 \|\mathbf{g}\|_{L^2}^2.$$

Thus, we obtain

$$(35) \quad \sum_{i=1}^4 |(B_i)| \leq \frac{C}{\mu} (\|\tilde{u}_2\|_{H^s}^2 + \|\tilde{v}_2\|_{H^s}^2) (\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{v}\|_{L^2}^2) + \mu \|\mathbf{u}\|_{\dot{H}^1}^2 + \mu \|\mathbf{v}\|_{\dot{H}^1}^2.$$

Combining estimates (34) and (35), and applying Grönwall's inequality, we obtain

$$\begin{aligned} \|\mathbf{u}(t, \cdot)\|_{L^2}^2 + \|\mathbf{v}(t, \cdot)\|_{L^2}^2 &\leq (\|\mathbf{u}(0, \cdot)\|_{L^2}^2 + \|\mathbf{v}(0, \cdot)\|_{L^2}^2) \times \\ &\quad \times \exp \left(C \int_0^t (\|\tilde{u}_1(\tau, \cdot)\|_{H^s} + \|\tilde{v}_1(\tau, \cdot)\|_{H^s} + \frac{1}{\mu} \|\tilde{u}_2(\tau, \cdot)\|_{H^s}^2 + \frac{1}{\mu} \|\tilde{v}_2(\tau, \cdot)\|_{H^s}^2) d\tau \right), \end{aligned}$$

where the integral converges since $\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2 \in \mathcal{C}_t H_x^s$. Thus, as $\mathbf{u}(0, \cdot) = 0$ and $\mathbf{v}(0, \cdot) = 0$, we conclude that $\tilde{u}_1 = \tilde{u}_2$, and $\tilde{v}_1 = \tilde{v}_2$. $\square$

Step 3: End of the proof of Theorem 1.1. Let u_0 be the initial datum satisfying assumptions (3) and (4).

On the one hand, using estimates (23) and (30) (where we write $\mathcal{H}u$ instead of $\tilde{v}$), together with the Picard iteration scheme applied to the fixed-point problem

$$(36) \quad u(t, \cdot) = K(t, \cdot) * u_0 - \int_0^t K(t - \tau, \cdot) * (u \partial_x u + \mathcal{H}u \partial_x (\mathcal{H}u))(\tau, \cdot) d\tau,$$

we obtain the existence of a time $T'_0 = T'_0(\|u_0\|_{H^s})$, verifying

$$(37) \quad T'_0(v_0) := \frac{1}{2} \frac{1}{(4C\|u_0\|_{H^s})^2},$$

and a solution $u \in \mathcal{C}([0, T'_0], H^s(\mathbb{R}))$ of equation (1).

On the other hand, considering the same initial datum u_0 , and defining $\tilde{u}_0 := u_0$, $\tilde{v}_0 := \mathcal{H}u_0$, by Proposition 3.1 there exists $(\tilde{u}, \tilde{v}) \in E_{\tilde{T}_0}$, a solution of system (19). Moreover, by the continuous embedding $E_{\tilde{T}_0} \subset \mathcal{C}([0, \tilde{T}_0], H^s(\mathbb{R}))$, we have $(\tilde{u}, \tilde{v}) \in \mathcal{C}([0, \tilde{T}_0], H^s(\mathbb{R}))$.

Finally, observe that $(u, \mathcal{H}u) \in \mathcal{C}([0, T'_0], H^s(\mathbb{R}))$ is also a solution of system (19). Defining

$$T_0 := \min(\tilde{T}_0, T'_0),$$

it follows from Proposition 3.2 that $\tilde{u} = u$ and $\tilde{v} = \mathcal{H}u$ on the interval $[0, T_0]$. Since $\tilde{u}$ and $\tilde{v}$ satisfy the pointwise decay estimates (20), the desired estimate (5) holds. This concludes the proof of Theorem 1.1.

3.2. Proof of Proposition 1.1. Arguing by contradiction, assume that (7) and (8) hold. Under these assumptions, for any fixed $0 < t \leq T_0$, we will prove that the solution $u(t, x)$ of equation (1) develops the following asymptotic profile with respect to the spatial variable x :

$$u(t, x) = \frac{1}{x} \Phi(M(u_0), t, u) + o(t) \left(\frac{1}{|x|} \right), \quad |x| \rightarrow +\infty,$$

where the expression $\Phi(M(u_0), t, u)$, defined in formula (54) below, does not depend on the variable x . This asymptotic profile yields a contradiction with the assumptions (7) and (8).

Returning to the mild formulation (36), we analyze its linear and nonlinear parts separately.

Lemma 3.3. *For the linear part, assuming (6), for any $t > 0$ the following asymptotic profile holds:*

$$(38) \quad K(t, \cdot) * u_0(x) = -\frac{\sin(\beta t)}{x} M(u_0) + o(t) \left(\frac{1}{|x|} \right), \quad |x| > 1,$$

where the quantity $M(u_0) \neq 0$ is defined in expression (6).

Proof. For fixed $t > 0$ and $|x| > 1$, we write

$$(39) \quad \begin{aligned} K(t, \cdot) * u_0(x) &= \int_{\mathbb{R}} K(t, x - y) u_0(y) dy \\ &= K(t, x) \int_{\mathbb{R}} u_0(y) dy + \int_{\mathbb{R}} (K(t, x - y) - K(t, x)) u_0(y) dy \\ &= K(t, x) M(u_0) + \int_{\mathbb{R}} (K(t, x - y) - K(t, x)) u_0(y) dy. \end{aligned}$$

For the first term, using the asymptotic profile (16), together with the fact that, by estimate (14), we have $I(t, x) = o(t) \left(\frac{1}{|x|} \right)$, it follows that

$$K(t, x)M(u_0) = -\frac{\sin(\beta t)}{x}, M(u_0) + I(t, x)M(u_0) = -\frac{\sin(\beta t)}{x} M(u_0) + o(t) \left(\frac{1}{|x|} \right), \quad |x| > 1.$$

For the second term, we have

$$(40) \quad \int_{\mathbb{R}} (K(t, x-y) - K(t, x)) u_0(y) dy = o(t) \left(\frac{1}{|x|} \right), \quad |x| > 1.$$

To prove this, we write

$$(41) \quad \begin{aligned} & \int_{\mathbb{R}} (K(t, x-y) - K(t, x)) u_0(y) dy \\ &= \int_{|y| \leq \frac{|x|}{2}} (K(t, x-y) - K(t, x)) u_0(y) dy + \int_{|y| > \frac{|x|}{2}} (K(t, x-y) - K(t, x)) u_0(y) dy \\ &= \underbrace{\int_{|y| \leq \frac{|x|}{2}} (K(t, x-y) - K(t, x)) u_0(y) dy}_{I_1(t, x)} + \underbrace{\int_{|y| > \frac{|x|}{2}} K(t, x-y) u_0(y) dy}_{I_2(t, x)} - \underbrace{K(t, x) \int_{|y| > \frac{|x|}{2}} u_0(y) dy}_{I_3(t, x)}, \end{aligned}$$

so that each term must be estimated separately.

To estimate the term $I_1(t, x)$, we use the following pointwise bound:

$$(42) \quad |\partial_x K(t, x)| \leq \frac{C}{x^2}, \quad t > 0, \quad x \neq 0,$$

for some constant $C > 0$ depending on μ . This estimate follows from arguments similar to those used in the proof of (12). For the reader's convenience, a detailed proof is provided in Appendix A.

Using (11) and well-known properties of the Fourier transform, we deduce that $K(t, \cdot) \in \mathcal{C}^1(\mathbb{R})$ for any $t > 0$. By the first-order Taylor expansion, for some $0 < \theta < 1$ we can write

$$(43) \quad K(t, x-y) - K(t, x) = -y \partial_x K(t, x-\theta y).$$

Moreover, since $|y| \leq \frac{|x|}{2}$, it follows that $|x-\theta y| \geq |x| - \theta|y| \geq |x| - |y| \geq \frac{|x|}{2}$. Therefore, using (42) we obtain

$$(44) \quad |\partial_x K(t, x-\theta y)| \leq \frac{C}{|x-\theta y|^2} \leq \frac{C}{x^2}.$$

With this estimate, and the fact that the initial datum satisfies (4) with $\gamma > 1$, the term $I_1(t, x)$ is controlled as

$$(45) \quad \begin{aligned} |I_1(t, x)| &\leq \int_{|y| \leq \frac{|x|}{2}} |K(t, x-y) - K(t, x)| |u_0(y)| dy \leq \int_{|y| \leq \frac{|x|}{2}} |y| |\partial_x K(t, x-\theta y)| \frac{C_0}{(1+|y|)^{1+\gamma}} dy \\ &\leq \frac{C}{x^2} \int_{\mathbb{R}} \frac{dy}{(1+|y|)^\gamma} \leq \frac{C}{x^2} = o\left(\frac{1}{|x|}\right). \end{aligned}$$

To estimate the term $I_2(t, x)$, note that since $|y| > \frac{|x|}{2}$ and $u_0(y)$ satisfies (4) (with $\gamma > 1$), by applying the Cauchy-Schwarz inequality and using the estimate (26) (with $p = 2$), we obtain

$$(46) \quad \begin{aligned} |I_2(t, x)| &\leq \int_{|y| > \frac{|x|}{2}} |K(t, x-y)| \frac{C_0}{(1+|y|)^{1+\gamma}} dy \leq \frac{C_0}{|x|^\gamma} \int_{\mathbb{R}} |K(t, x-y)| \frac{1}{1+|y|} dy \\ &\leq \frac{C_0}{|x|^\gamma} \|K(t, \cdot)\|_{L^2} \left\| \frac{1}{1+|y|} \right\|_{L^2} \leq \frac{C}{|x|^\gamma} \frac{\eta(t)}{t^{\frac{1}{2}}} = o(t) \left(\frac{1}{|x|} \right). \end{aligned}$$

Finally, to estimate the term $I_3(t, x)$, we use the asymptotic profile for $K(t, x)$ given in (16), recalling that $I(t, x) = o(t) \left(\frac{1}{|x|} \right)$. Moreover, since $u_0(y)$ satisfies (4) (with $\gamma > 1$) we have $u_0 \in L^1(\mathbb{R})$. By the standard dominated convergence theorem, it follows that

$$\lim_{|x| \rightarrow +\infty} \int_{|y| > \frac{|x|}{2}} |u_0(y)| dy = 0.$$

Therefore,

$$(47) \quad |I_3(t, x)| \leq \left(\frac{|\sin(\beta t)|}{|x|} + |I(t, x)| \right) \int_{|y| > \frac{|x|}{2}} |u_0(y)| dy = o(t) \left(\frac{1}{|x|} \right).$$

Having proved estimates (45), (46), and (47), it follows from identity (41) together with equality (40) that the desired profile (38) holds. $\square$

Lemma 3.4. *For the nonlinear part, assuming (7) and (8), we obtain that for any $0 < t \leq T_0$ (where the time $T_0 > 0$ is given in Theorem 1.1), the following asymptotic profile holds:*

$$(48) \quad \begin{aligned} & \int_0^t K(t - \tau, \cdot) * (u \partial_x u + \mathcal{H}u \partial_x(\mathcal{H}u))(\tau, \cdot) d\tau \\ &= -\frac{1}{x} \left[\int_0^t \sin(\beta(t - \tau)) \left(\int_{\mathbb{R}} (u \partial_y u + \mathcal{H}u \partial_y(\mathcal{H}u))(\tau, y) dy \right) d\tau \right] + o(t) \left(\frac{1}{|x|} \right), \end{aligned} \quad |x| > 2M_\varepsilon.$$

Proof. Note that it is enough to study the first term $\int_0^t K(t - \tau, \cdot) * (u \partial_x u)(\tau, \cdot) d\tau$, since, by assumptions (7) and (8), the second term $\int_0^t K(t - \tau, \cdot) * (\mathcal{H}u \partial_x(\mathcal{H}u))(\tau, \cdot) d\tau$ can be treated by the same arguments.

Following the same decomposition used in identities (39) and (40), for fixed $0 < t \leq T_0$ and $|x| > 2M_\varepsilon$, we write

$$(49) \quad \begin{aligned} & \int_0^t \int_{\mathbb{R}} K(t - \tau, x - y) u(\tau, y) \partial_y u(\tau, y) dy d\tau \\ &= \underbrace{\int_0^t K(t - \tau, x) \left(\int_{\mathbb{R}} u(\tau, y) \partial_y u(\tau, y) dy \right) d\tau}_{I_1(t, x)} \\ & \quad + \underbrace{\int_0^t \int_{|y| \leq \frac{|x|}{2}} (K(t - \tau, x - y) - K(t - \tau, x)) u(\tau, y) \partial_y u(\tau, y) dy d\tau}_{I_2(t, x)} \\ & \quad + \underbrace{\int_0^t \int_{|y| > \frac{|x|}{2}} K(t - \tau, x - y) u(\tau, y) \partial_y u(\tau, y) dy d\tau}_{I_3(t, x)} \\ & \quad - \underbrace{\int_0^t K(t - \tau, x) \left(\int_{|y| > \frac{|x|}{2}} u(\tau, y) \partial_y u(\tau, y) dy \right) d\tau}_{I_4(t, x)}, \end{aligned}$$

where each term on the right-hand side must be analyzed separately. Without loss of generality, we may assume that $2M_\varepsilon > 1$.

For the first term, using the identity (16), we obtain

$$I_1(t, x) = -\frac{1}{x} \int_0^t \sin(\beta(t - \tau)) \left(\int_{\mathbb{R}} u(\tau, y) \partial_y u(\tau, y) dy \right) d\tau + \int_0^t I(t - \tau, x) \left(\int_{\mathbb{R}} u(\tau, y) \partial_y u(\tau, y) dy \right) d\tau,$$

where, for the second expression above, using estimate (14), the Cauchy-Schwarz inequality, and the continuous embeddings $H^s(\mathbb{R}) \subset L^2(\mathbb{R})$ and $H^s(\mathbb{R}) \subset \dot{H}^1(\mathbb{R})$, since $s > \frac{3}{2}$, it follows that

$$\begin{aligned} & \left| \int_0^t K(t-\tau, x) \left(\int_{\mathbb{R}} u(\tau, y) \partial_y u(\tau, y) dy \right) d\tau \right| \leq \frac{C}{x^2} \int_0^t (t-\tau)^{\frac{1}{2}} \|u(\tau, \cdot)\|_{L^2} \|u(\tau, \cdot)\|_{\dot{H}^1} d\tau \\ & \leq \frac{C t^{\frac{3}{2}}}{x^2} \left(\sup_{0 \leq \tau \leq T_0} \|u(\tau, \cdot)\|_{H^s}^2 \right). \end{aligned}$$

Therefore, we obtain

$$(50) \quad I_1(t, x) = -\frac{1}{x} \int_0^t \sin(\beta(t-\tau)) \left(\int_{\mathbb{R}} u(\tau, y) \partial_y u(\tau, y) dy \right) d\tau + o(t) \left(\frac{1}{|x|} \right).$$

For the second term, from identity (43) and estimate (44), applying the Cauchy-Schwarz inequality together with assumption (8), it follows that

$$\begin{aligned} (51) \quad I_2(t, x) & \leq \frac{C}{x^2} \int_0^t \int_{|y| \leq \frac{|x|}{2}} |y| |u(\tau, y)| |\partial_y u(\tau, y)| dy d\tau \leq \frac{C}{x^2} \int_0^t \int_{\mathbb{R}} |y| |u(\tau, y)| |\partial_y u(\tau, y)| dy d\tau \\ & \leq \frac{C t}{x^2} \left(\sup_{0 \leq \tau \leq T_0} \| |x| u(\tau, \cdot) \|_{L^2} \right) \left(\sup_{0 \leq \tau \leq T_0} \|u(\tau, \cdot)\|_{H^s} \right) \\ & = o(t) \left(\frac{1}{|x|} \right). \end{aligned}$$

For the third term, by assumption (7) and the fact that $|y| \geq \frac{|x|}{2} \geq M_\varepsilon$, together with the Cauchy-Schwarz inequality and estimate (26), we have

$$\begin{aligned} (52) \quad I_3(t, x) & \leq \int_0^t \int_{|y| > \frac{|x|}{2}} |K(t-\tau, x-y)| \frac{C_2}{\tau^{\frac{1}{2}} |y|^{1+\varepsilon}} |\partial_y u(\tau, y)| dy \\ & \leq \frac{C_2}{|x|^{1+\varepsilon}} \int_0^t \frac{1}{\tau^{\frac{1}{2}}} \left(\int_{\mathbb{R}} |K(t-\tau, x-y)| |\partial_y u(\tau, y)| dy \right) d\tau \\ & \leq \frac{C_2}{|x|^{1+\varepsilon}} \int_0^t \frac{1}{\tau^{\frac{1}{2}}} \|K(t-\tau)\|_{L^2} \|\partial_y u(\tau, \cdot)\|_{L^2} d\tau \\ & \leq \frac{C}{|x|^{1+\varepsilon}} \left(\int_0^t \frac{d\tau}{\tau^{\frac{1}{2}} (t-\tau)^{\frac{1}{2}}} \right) \left(\sup_{0 \leq \tau \leq T_0} \|u(\tau, \cdot)\|_{H^s} \right) \\ & \leq \frac{C}{|x|^{1+\varepsilon}} \left(\sup_{0 \leq \tau \leq T_0} \|u(\tau, \cdot)\|_{H^s} \right) \\ & = o \left(\frac{1}{|x|} \right). \end{aligned}$$

Finally, since $u(\tau, \cdot) \partial_y u(\tau, \cdot) \in L^1(\mathbb{R})$ for all $0 < \tau < t$, similar arguments as in (47) yield

$$(53) \quad I_4(t, x) = o(t) \left(\frac{1}{|x|} \right).$$

Gathering the identities and estimates (50), (51), (52), and (53) into identity (49), the desired asymptotic profile (48) follows. $\square$

End of the proof of Proposition 1.1. Returning to the mild formulation (36) and using the asymptotic profiles (38) and (48), for any $0 < t \leq T_0$ and $|x| > 2M_\varepsilon$, the solution $u(t, x)$ satisfies

$$u(t, x) = \frac{1}{x} \left[-\sin(\beta t) M(u_0) - \int_0^t \sin(\beta(t-\tau)) \left(\int_{\mathbb{R}} (u \partial_y u + \mathcal{H}u \partial_y (\mathcal{H}u))(\tau, y) dy \right) d\tau \right] + o(t) \left(\frac{1}{|x|} \right),$$

where, defining

$$(54) \quad \Phi(M(u_0), t, u) =: -\sin(\beta t)M(u_0) - \int_0^t \sin(\beta(t-\tau)) \left(\int_{\mathbb{R}} (u \partial_y u + \mathcal{H}u \partial_y(\mathcal{H}u))(\tau, y) dy \right) d\tau,$$

we can write

$$u(t, x) = \frac{1}{x} \Phi(M(u_0), t, u) + o(t) \left(\frac{1}{|x|} \right), \quad |x| > 2M_\varepsilon.$$

From this profile of $u(t, x)$, we can derive a contradiction, which invalidates both assumptions (7) and (8).

We write

$$|u(t, x)| = \left| \frac{1}{x} \Phi(M(u_0), t, u) - \left(-o(t) \left(\frac{1}{|x|} \right) \right) \right| \geq \frac{1}{|x|} |\Phi(M(u_0), t, u)| - \left| o(t) \left(\frac{1}{|x|} \right) \right|.$$

By the definition of the term $o(t) \left(\frac{1}{|x|} \right)$, for $|\Phi(M(u_0), t, u)| > 0$ there exists $M_t > 0$ sufficiently large so that $\left| o(t) \left(\frac{1}{|x|} \right) \right| < \frac{1}{2|x|} |\Phi(M(u_0), t, u)|$, for any $|x| > M_t$. Hence, for $|x| > \max(2M_\varepsilon, M_t)$, we obtain

$$(55) \quad \frac{1}{2|x|} |\Phi(M(u_0), t, u)| \leq |u(t, x)|.$$

Fixing the time $t = \frac{T_0}{2}$, assumption (7) yields

$$\frac{1}{2} \left| \Phi \left(M(u_0), \frac{T_0}{2}, u \right) \right| \frac{1}{|x|} \leq \frac{C_2}{\left(\frac{T_0}{2} \right)^{\frac{1}{2}} |x|^{1+\varepsilon}}, \quad |x| > \max(2M_\varepsilon, M_{\frac{T_0}{2}}),$$

which gives us a contradiction by letting $|x| \rightarrow +\infty$.

On the other hand, from (55) it follows that

$$\frac{1}{2} |\Phi(M(u_0), t, u)| \leq |x| |u(t, x)|, \quad |x| > \max(2M_\varepsilon, M_t),$$

and hence, for any fixed time $0 < t \leq T_0$, we obtain $u(t, \cdot) \notin L^2(\mathbb{R}, |x| dx)$, which contradicts assumption (8). Proposition 1.1 has now been proven.

4. GLOBAL IN TIME PROPERTIES

4.1. Proof of Proposition 1.2. From equation (1), we get:

$$\frac{1}{2} \frac{d}{dt} \|u(t, \cdot)\|_{L^2}^2 = - \int_{\mathbb{R}} (\mathcal{H}u) \partial_x(\mathcal{H}u) u dx - \mu \|u(t, \cdot)\|_{H^1}^2 \leq - \int_{\mathbb{R}} (\mathcal{H}u) \partial_x(\mathcal{H}u) u dx.$$

To control the term on the right-hand-side, we write:

$$- \int_{\mathbb{R}} (\mathcal{H}u) \partial_x(\mathcal{H}u) u dx = \frac{1}{2} \int_{\mathbb{R}} (\mathcal{H}u)^2 \partial_x u dx \leq \|\partial_x u\|_{L^\infty} \|u\|_{L^2}^2,$$

and applying Grönwall's inequality, we obtain:

$$(56) \quad \|u(t, \cdot)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 \exp \left(\int_0^t \|\partial_x u(\tau, \cdot)\|_{L^\infty} d\tau \right).$$

Using this estimate, we establish the blow-up criterion (9) as follows. First, assume that

$$(57) \quad \lim_{t \rightarrow T_*} \|u(t, \cdot)\|_{H^s} = +\infty.$$

Arguing by contradiction, suppose that

$$(58) \quad \int_0^{T_*} \|\partial_x u(t, \cdot)\|_{L^\infty} dt < +\infty.$$

Define

$$M_0 := \|u_0\|_{L^2}^2 \exp \left(\int_0^{T_*} \|\partial_x u(t, \cdot)\|_{L^\infty} dt \right) < +\infty.$$

Then, from estimate (56), for all $0 < t \leq T_*$ we obtain

$$(59) \quad \|u(t, \cdot)\|_{L^2}^2 \leq M_0.$$

Now, recall that for any initial datum $v_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, the existence time $T_0 = T_0(v_0) > 0$ of the corresponding solution $v \in \mathcal{C}([0, T_0], H^s(\mathbb{R}))$ to equation (1) is given by the expression (37):

$$T_0(v_0) := \frac{1}{2} \frac{1}{(4C\|v_0\|_{H^s})^2} \leq \frac{1}{2} \frac{1}{(4C\|v_0\|_{L^2})^2}.$$

Consequently, the time $T_0(v_0)$ is a decreasing function of $\|v_0\|_{L^2}^2$. This property implies that there exists a time $0 < T_1 < T_*$ such that for any initial datum satisfying $\|v_0\|_{L^2}^2 \leq M_0$, the associated solution $v \in \mathcal{C}_t H_x^s$ exists at least on the interval $[0, T_1]$, and satisfies $v \in \mathcal{C}([0, T_1], H^s(\mathbb{R}))$.

In this context, fix $0 < \varepsilon < T_1$, and consider the initial datum $v_0 := u(T_* - \varepsilon, \cdot)$, which satisfies $\|v_0\|_{L^2}^2$ by the bound in (59). As mentioned above, the corresponding solution $v(t, x)$ is defined over the interval $[0, T_1]$.

Therefore, gathering the solutions $u(t, x)$ and $v(t, x)$, which arise from the initial datum u_0 and v_0 respectively, we define the function

$$\tilde{u}(t, \cdot) = \begin{cases} u(t, \cdot), & \text{for } 0 \leq t \leq T_* - \varepsilon, \\ v(t, \cdot), & \text{for } T_* - \varepsilon \leq t \leq T_* - \varepsilon + T_1. \end{cases}$$

Since $0 < \varepsilon < T_1$ it follows that $T_* - \varepsilon + T_1 > T_*$, implying that $\tilde{u}(t, \cdot)$ is a solution of equation (1) with initial datum u_0 , defined on the extended interval $[0, T_* - \varepsilon + T_1]$, satisfies $\tilde{u} \in \mathcal{C}([0, T_* - \varepsilon + T_1], H^s(\mathbb{R}))$. This contradicts assumption (57).

Now, suppose that

$$(60) \quad \int_0^{T_*} \|\partial_x u(t, \cdot)\|_{L^\infty} dt = +\infty.$$

Again, arguing by contradiction, assume that

$$\lim_{t \rightarrow T_*} \|u(t, \cdot)\|_{H^s} < +\infty.$$

Then, the solution $u(t, x)$ of equation (1) can be extended beyond time T_* , and there exists an $\varepsilon > 0$ such that $u \in \mathcal{C}([0, T_* + \varepsilon], H^s(\mathbb{R}))$.

Since $s > \frac{3}{2}$, applying the Sobolev embedding yields

$$\int_0^{T_*} \|\partial_x u(t, \cdot)\|_{L^\infty} dt \leq T_* \left(\sup_{0 \leq t \leq T_* + \varepsilon} \|u(t, \cdot)\|_{H^s} \right) < +\infty,$$

which contradicts (60). Therefore, Proposition 1.2 is proven.

4.2. Proof of Proposition 1.3. Assume that (10) holds. Using this fact, we will show that the pointwise estimate (5) holds for all times $0 < t \leq T_*$. Specifically, define

$$g(t) := t^{\frac{1}{2}} \left(\|(1 + |x|)^{\min(1, \gamma)} u(t, \cdot)\|_{L^\infty} + \|(1 + |x|)^{\min(1, \gamma)} \mathcal{H}u(t, \cdot)\|_{L^\infty} \right),$$

and we will show that $g(t) < +\infty$ for every $0 < t \leq T_*$.

Using the mild formulation of the solution $u(t, x)$ given in (36), we have

$$u(t, \cdot) = K(t, \cdot) * u_0 - \int_0^t K(t - \tau, \cdot) * (u \partial_x u + \mathcal{H}u \partial_x (\mathcal{H}u))(\tau, \cdot) d\tau,$$

and

$$\mathcal{H}u(t, \cdot) = \mathcal{H}K(t, \cdot) * u_0 - \int_0^t \mathcal{H}K(t - \tau, \cdot) * (u \partial_x u + \mathcal{H}u \partial_x (\mathcal{H}u))(\tau, \cdot) d\tau.$$

Since $u(t, \cdot)$ and $\mathcal{H}u(t, \cdot)$, as well as $K(t, \cdot)$ and $\mathcal{H}K(t, \cdot)$, satisfy the same estimates, we introduce the unified notation

$$(61) \quad \mathbf{u}(t, \cdot) = \mathbf{K}(t, \cdot) * u_0 - \int_0^t \mathbf{K}(t - \tau, \cdot) * \mathbf{u} \partial_x \mathbf{u}(\tau, \cdot) d\tau,$$

where $\mathbf{u} \in \{u, \mathcal{H}u\}$ and $\mathbf{K} \in \{K, \mathcal{H}K\}$, for simplicity of presentation.

With a slight abuse of notation, we rewrite

$$(62) \quad g(t) := t^{\frac{1}{2}} \|(1 + |x|)^{\min(1, \gamma)} \mathbf{u}(t, \cdot)\|_{L^\infty}.$$

Using the integral formulation above, we obtain

$$(63) \quad g(t) \leq \underbrace{t^{\frac{1}{2}} \|(1 + |x|)^{\min(1, \gamma)} \mathbf{K}(t, \cdot) * u_0\|_{L^\infty}}_{g_1(t)} + \underbrace{t^{\frac{1}{2}} \left\| (1 + |x|)^{\min(1, \gamma)} \int_0^t \mathbf{K}(t - \tau, \cdot) * \mathbf{u} \partial_x \mathbf{u}(\tau, \cdot) d\tau \right\|_{L^\infty}}_{g_2(t)},$$

where each term on the right-hand side must be treated separately.

For the first term, by estimate (28), for any $0 < t \leq T_*$ we directly have

$$(64) \quad g_1(t) \leq C \eta(T_*) (\|u_0\|_{L^1} + C_0) =: \mathfrak{C}_0.$$

For the second term, using the decomposition as in (31), we obtain

$$(65) \quad \begin{aligned} & \int_0^t \mathbf{K}(t - \tau, \cdot) * \mathbf{u} \partial_x \mathbf{u}(\tau, x) d\tau \\ &= \underbrace{\int_0^t \int_{|y| \leq \frac{|x|}{2}} \mathbf{K}(t - \tau, x - y) \mathbf{u} \partial_x \mathbf{u}(\tau, y) dy d\tau}_{I_1(t, x)} + \underbrace{\int_0^t \int_{|y| > \frac{|x|}{2}} \mathbf{K}(t - \tau, x - y) \mathbf{u} \partial_x \mathbf{u}(\tau, y) dy d\tau}_{I_2(t, x)}. \end{aligned}$$

The term $I_1(t, x)$ has already been estimated in (32) (with $\mathbf{u}$ in place of $\tilde{u}$ and $\tilde{v}$), yielding for $0 < t \leq T_*$,

$$(66) \quad \begin{aligned} I_1(t, x) &\leq C \frac{\eta(T_*)}{1 + |x|} \left(\int_0^t \frac{d\tau}{(t - \tau)^{\frac{1}{2}}} \right) \sup_{0 \leq \tau \leq t} \|\mathbf{u}(\tau, \cdot)\|_{H^s}^2 \\ &\leq C \frac{\eta(T_*) T_*^{\frac{1}{2}}}{(1 + |x|)^{\min(1, \gamma)}} \left(\sup_{0 \leq \tau \leq T_*} \|\mathbf{u}(\tau, \cdot)\|_{H^s}^2 \right) \\ &=: \frac{\mathfrak{C}_1}{(1 + |x|)^{\min(1, \gamma)}}. \end{aligned}$$

For the term $I_2(t, x)$, using $g(t)$ as defined in (62), we write

$$\begin{aligned} I_2(t, x) &\leq \int_0^t \int_{|y| > \frac{|x|}{2}} |\mathbf{K}(t - \tau, x - y)| \frac{\tau^{\frac{1}{2}} (1 + |y|)^{\min(1, \gamma)} |\mathbf{u}(\tau, y)|}{\tau^{\frac{1}{2}} (1 + |y|)^{\min(1, \gamma)}} |\partial_y \mathbf{u}(\tau, y)| \\ &\leq \frac{C}{(1 + |x|)^{\min(1, \gamma)}} \int_0^t \frac{g(\tau)}{\tau^{\frac{1}{2}}} \int_{\mathbb{R}} |\mathbf{K}(t - \tau, x - y)| |\partial_y \mathbf{u}(\tau, y)| dy d\tau. \end{aligned}$$

Applying the Cauchy–Schwarz inequality and the estimate (26) with $p = 2$, we obtain

$$\begin{aligned} (67) \quad I_2(t, x) &\leq \frac{C}{(1 + |x|)^{\min(1, \gamma)}} \int_0^t \frac{g(\tau)}{\tau^{\frac{1}{2}}} \|\mathbf{K}(t - \tau, \cdot)\|_{L^2} \|\partial_y \mathbf{u}(\tau, \cdot)\|_{L^2} d\tau \\ &\leq \frac{C \eta(T_*)}{(1 + |x|)^{\min(1, \gamma)}} \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} g(\tau) \|\mathbf{u}(\tau, \cdot)\|_{H^s} d\tau \\ &\leq \frac{C \eta(T_*)}{(1 + |x|)^{\min(1, \gamma)}} \left(\sup_{0 \leq \tau \leq T_*} \|\mathbf{u}(\tau, \cdot)\|_{H^s} \right) \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} g(\tau) d\tau \\ &=: \frac{\mathfrak{C}_2}{(1 + |x|)^{\min(1, \gamma)}} \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} g(\tau) d\tau. \end{aligned}$$

Combining estimates (66) and (67) in (65), we obtain, for $0 < t \leq T_*$,

$$\begin{aligned} (68) \quad g_2(t) &\leq t^{\frac{1}{2}} \mathfrak{C}_1 + t^{\frac{1}{2}} \mathfrak{C}_2 \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} g(\tau) d\tau \leq T_*^{\frac{1}{2}} \mathfrak{C}_1 + t^{\frac{1}{2}} \mathfrak{C}_2 \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} g(\tau) d\tau \\ &=: \mathfrak{C}_3 + t^{\frac{1}{2}} \mathfrak{C}_2 \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} g(\tau) d\tau. \end{aligned}$$

Finally, combining (64) and (68) in (63), we conclude that

$$g(t) \leq \mathfrak{C}_0 + \mathfrak{C}_3 + t^{\frac{1}{2}} \mathfrak{C}_2 \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} g(\tau) d\tau =: \mathfrak{C}_4 + t^{\frac{1}{2}} \mathfrak{C}_2 \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} g(\tau) d\tau.$$

Once we have this estimate for $g(t)$, we invoke the following Grönwall-type inequality. For a proof, see [5, Lemma 3.4].

Lemma 4.1. *Let $T > 0$ and $g : [0, T] \rightarrow \mathbb{R}$ be a non-negative and locally bounded function, which for any $0 < t \leq T$, satisfies:*

$$(69) \quad g(t) \leq M_1 + M_2 \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} g(\tau) d\tau,$$

for some constants $M_1, M_2 > 0$ possibly depending on T . Additionally, define

$$M := \int_0^1 \frac{d\tau}{(1 - \tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}}.$$

If $M_2 < \frac{1}{M}$, then it follows that $g(t) \leq M_1$ for every $0 < t \leq T$.

Within the framework of this lemma, we choose a time $0 < T_1 \leq T_*$, and set $M_1 = \mathfrak{C}_4$ and $M_2 = T_1^{\frac{1}{2}} \mathfrak{C}_2$. Thus, for any $0 < t \leq T_1$, the function $g(t)$ defined in (62) satisfies

$$g(t) \leq \mathfrak{C}_4 + T_1^{\frac{1}{2}} \mathfrak{C}_2 \int_0^t \frac{1}{(t - \tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} g(\tau) d\tau,$$

which corresponds to the estimate (69).

We then fix $0 < T_1 \leq T_*$ sufficiently small so that $T_1^{\frac{1}{2}} \mathfrak{C}_2 < \frac{1}{M}$. Consequently, it follows that $g(t) \leq \mathfrak{C}_4$ for every $0 < t \leq T_1$.

Thereafter, we can iterate this process up to the time T_* as follows. We now consider the initial datum $u(T_1, \cdot)$, which, by assumption (10), satisfies $\mathbf{u}(T_1, \cdot) \in L^1(\mathbb{R})$.

Remark 2. *Returning to the mild formulation (61), recall from Remark 1 that $\mathbf{K}(t, \cdot) \notin L^1(\mathbb{R})$. Consequently, even assuming that $u_0 \in L^1(\mathbb{R})$, it is not clear whether this information allows us to deduce that $\mathbf{K}(t, \cdot) * u_0 \in L^1(\mathbb{R})$. Therefore, to the best of our knowledge, it remains open whether the L^1 -integrability of the initial data persists for the associated solution.*

For any time $t \geq T_1$, we then consider the fixed-point equation

$$\mathbf{u}(t, \cdot) = K(t - T_1, \cdot) * \mathbf{u}(T_1, \cdot) - \int_{T_1}^t K(t - \tau, \cdot) * \mathbf{u} \partial_x \mathbf{u}(\tau, \cdot) d\tau.$$

Since $\mathbf{u}(T_1, \cdot) \in H^s \cap L^1(\mathbb{R})$, estimates (64) and (68) remain valid. Using again Lemma 4.1, we obtain, for some constant $\mathfrak{C}'_4 > 0$ and for a sufficiently small time $T_2 > 0$, that $g(t) \leq \mathfrak{C}'_4$ for any $T_1 \leq t \leq T_2$.

Finally, by repeating this procedure a finite number of times, we reach the time T_* . This concludes the proof of Proposition 1.3.

APPENDIX A. PROOF OF ESTIMATE (42)

Using well-known properties of the Fourier transform, along with identity (11) and the fact that $\partial_\xi e^{2\pi i x \xi} = 2\pi i x e^{2\pi i x \xi}$, integrating by parts we write

$$\begin{aligned} \partial_x K(t, x) &= \int_{-\infty}^0 e^{2\pi i x \xi} i \xi e^{-\mu \xi^2 t - i \beta t} d\xi + \int_0^{+\infty} e^{2\pi i x \xi} i \xi e^{-\mu \xi^2 t + i \beta t} d\xi \\ &= \frac{1}{2\pi i x} \int_{-\infty}^0 \partial_\xi (e^{2\pi i x \xi}) i \xi e^{-\mu \xi^2 t - i \beta t} d\xi + \frac{1}{2\pi i x} \int_0^{+\infty} \partial_\xi (e^{2\pi i x \xi}) i \xi e^{-\mu \xi^2 t + i \beta t} d\xi \\ &= -\frac{1}{2\pi x} \int_{-\infty}^0 e^{2\pi i x \xi} e^{-\mu \xi^2 t - i \beta t} (1 - 2\mu \xi^2 t) d\xi - \frac{1}{2\pi x} \int_0^{+\infty} e^{2\pi i x \xi} e^{-\mu \xi^2 t + i \beta t} (1 - 2\mu \xi^2 t) d\xi. \end{aligned}$$

Note that, in contrast with identity (13), the symbol $i\xi$ (corresponding to the derivative ∂_x) cancels all remaining terms at $\xi = 0^-$ and $\xi = 0^+$. Consequently, we can repeat the same argument to obtain

$$\begin{aligned} \partial_x K(t, x) &= -\frac{1}{4\pi^2 i x^2} \int_{-\infty}^0 \partial_\xi (e^{2\pi i x \xi}) e^{-\mu \xi^2 t - i \beta t} (1 - 2\mu \xi^2 t) d\xi - \frac{1}{4\pi^2 i x^2} \int_0^{+\infty} \partial_\xi (e^{2\pi i x \xi}) e^{-\mu \xi^2 t + i \beta t} (1 - 2\mu \xi^2 t) d\xi \\ &= \frac{1}{4\pi^2 i x^2} \left(\int_{-\infty}^0 (e^{2\pi i x \xi}) \partial_\xi (e^{-\mu \xi^2 t - i \beta t} (1 - 2\mu \xi^2 t)) d\xi + (e^{2\pi i x \xi}) (e^{-\mu \xi^2 t - i \beta t} (1 - 2\mu \xi^2 t)) \Big|_{-\infty}^0 \right) \\ &\quad + \frac{1}{4\pi^2 i x^2} \left(\int_0^{+\infty} (e^{2\pi i x \xi}) \partial_\xi (e^{-\mu \xi^2 t + i \beta t} (1 - 2\mu \xi^2 t)) d\xi + (e^{2\pi i x \xi}) (e^{-\mu \xi^2 t + i \beta t} (1 - 2\mu \xi^2 t)) \Big|_0^{+\infty} \right) \\ &= -\frac{1}{4\pi^2 i x^2} \left(\int_{-\infty}^0 e^{2\pi i x \xi} e^{-\mu \xi^2 t - i \beta t} (4\mu^2 \xi^3 t^2 - 6\mu \xi t) d\xi + \int_0^{+\infty} e^{2\pi i x \xi} e^{-\mu \xi^2 t + i \beta t} (4\mu^2 \xi^3 t^2 - 6\mu \xi t) d\xi \right) \\ &\quad + \frac{e^{-i\beta t} - e^{i\beta t}}{4\pi^2 i x^2} \\ &=: -\frac{1}{4\pi^2 i x^2} J(t, x) - \frac{\sin(\beta t)}{4\pi^2 i x^2}. \end{aligned}$$

As before, using the rapid decay of $e^{-\mu \xi^2 t}$, we obtain

$$|J(t, x)| \leq \int_{-\infty}^{+\infty} e^{-\mu |\sqrt{t} \xi|^2} \left(4\mu^2 |\sqrt{t} \xi|^3 + 6\mu |\sqrt{t} \xi| \right) \sqrt{t} d\xi \leq C < +\infty,$$

for some constant $C > 0$ depending on μ . Hence, for $x \neq 0$, the desired estimate (42) follows.

Acknowledgements. All the authors were partially supported by MathAmSud WAFFLE 23-MATH-18.

Statements and Declaration. Data sharing does not apply to this article as no datasets were generated or analyzed during the current study. In addition, the authors declare that they have no conflicts of interest, and all of them have equally contributed to this paper.

REFERENCES

- [1] D. Bekiranov. *The initial-value problem for the generalized Burgers equation*. Differ. Int. Eq. 9, 125-1265 (1996).
- [2] C. Bjorland, L. Brandolese, D. Iftimie and M.E. Schonbek. *L^p solutions of the steady-state Navier-Stokes equations with rough external forces*. Comm. Part. Diff. Equ., 36, 216 - 246 (2011).
- [3] L. Brandolese and Y. Meyer. *On the instantaneous spreading for the Navier-Stokes system in the whole space*. Contr. Optim. Calc. Var. 8, pp. 273–285 (2002).
- [4] L. Brandolese, F. Vigneron. *New asymptotic profiles of nonstationnary solutions of the Navier-Stokes system* J. Math. Pures Appl. 88,64–86 (2007).
- [5] L. Brandolese and G. Karch. *Far field asymptotics of solutions to convection equation with anomalous diffusion*. J. Evolution Equations. 8: 307–326 (2008).
- [6] L. Brandolese. *Fine properties of self-similar solutions of the Navier-Stokes equations*. Arch. Rational Mech. Anal., 192, N.3, 375–401 (2009).
- [7] A. Constantin and L. Molinet. *Global Existence and Finite-Time Blow-Up for a Nonlinear Nonlocal Evolution Equation*. Commun. Math. Phys. 402, 3233–3252 (2023).
- [8] M.F. Cortez and O. Jarrin. *On decay properties and asymptotic behavior of solutions to a non-local perturbed KdV equation*. Nonlinear Anal. 187, 365–396 (2019).
- [9] M.F. Cortez and O. Jarrin. *Spatial behavior of solutions for a large class of non-local pde's arising from stratified flows*. Differential Integral Equations, 34 (9/10) 539 - 594 (2021).
- [10] M.F. Cortez and O. Jarrin. *Sharp well-posedness and spatial decaying for a generalized dispersive-dissipative Kuramoto-type equation and applications to related models*. arXiv:2308.01449 to appear in Advances in Differential Equations (2025).
- [11] D.B. Dix. *Nonuniqueness and uniqueness in the initial-value problem for Burgers equation*. SIAM J. Math. Anal. 27, 708–724 (1996).
- [12] S. Dobrokhotov and A. Shafarevich. *Some integral identities and remarks on the decay at infinity of the solutions to the Navier-Stokes equations in the entire space*. Russ. J. Math. Phys., 2:133-135 (1994).
- [13] G. Fonseca, F. Linares and G. Ponce. *On persistence properties in fractional weighted spaces*. Proc. Amer. Math. Soc. 143, 5353-5367 (2015).
- [14] A. Muñoz. *On decay of solutions to some perturbations of the Korteweg-de Vries equation*. Nonlinear Differ. Equ. Appl. 31, 79 (2024).
- [15] J. Nahas and G. Ponce. *On the well-posedness of the modified Korteweg-de Vries equation in weighted Sobolev spaces* Preprint available at <https://web.math.ucsb.edu/~ponce/jn-gp-2.pdf> (2010).

E-mail address: oscar.jarrin@udla.edu.ec

E-mail address: manuel.cortez@epn.edu.ec